Dynamical Systems

Let $S = \mathbb{R}$ or $\mathbb{C}$, and let $f \colon S \to S$. The *filled Julia set* for f is

$$K(f) = \{z \in S \colon \mathrm{Orb}_f(z) \text{ is bounded}\}.$$

Thus, $K(f)$ is the set of points $z \in S$ whose iterates are bounded: there exists a real number r such that $|f^n(z)| \leq r$ for all $n \geq 0$. The *Julia set*, denoted $J(f)$, is the boundary[1] of $K(f)$.

Example. Let $f \colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 - 1$. We saw last time that $K(f) = [-\alpha, \alpha]$ where $\alpha = \frac{1+\sqrt{5}}{2}$. Thus, $J(f)$ consists of the two endpoints: $J(f) = \{-\alpha, \alpha\}$.

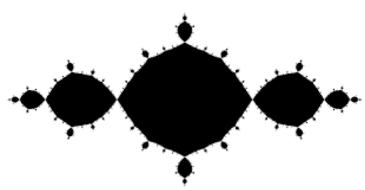
PROBLEM 1. What are the filled Julia set and the Julia set for the function $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$?

SOLUTION: We have $K(f) = [-1, 1]$, and $J(f) = \{-1, 1\}$.

PROBLEM 2. What are the filled Julia set and the Julia set for the function $f \colon \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$?

SOLUTION: We have $K(f)$ is the closed disc of radius one centered at the origin, and $J(f)$ is the circle of radius one centered at the origin.

PROBLEM 3. Consider the filled Julia set $K(f)$ for the function $f \colon \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2 - 1$. The set $K(f) \subseteq \mathbb{C}$ is pictured below:



(a) What is the horizontal line segment running through the middle (along the real axis)?

(b) Is $i \in K(f)$? What about $i/2$? (Hint: use that fact that you know something about the filled Julia set for f restricted to the real numbers. Note: $1.61 \leq (1 + \sqrt{5})/2 \leq 1.62$.)

SOLUTION:

(a) $[-\alpha, \alpha]$ with α as defined above.

[1] If X is a subset of a topological space, the *closure* of X, denoted $\overline{X}$, is the smallest closed set containing K. It is the intersection of all closed set containing X. The *boundary* of X is the intersection of the closure of X and the closure of the complement of X. Example: the closure of an open ball in $\mathbb{C}$ is a circle.

(b) We have $f(i) = i^2 - 1 = -2 < -\alpha$. From this iterate onward, we may as well be considering the real version of f with filled in Julia set $[-\alpha, \alpha]$. Since $2 \notin [-\alpha, \alpha]$, its iterates are not bounded. Hence, $i \notin K(f)$. On the other hand, $f(i/2) = (i/2)^2 - 1 = -5/4 = -1.25 \in [-\alpha, \alpha]$. Hence, its iterates are bounded. Therefore, $i/2 \in K(f)$.

Problem 4. Let $c \in \mathbb{C}$ and consider the function $f_c \colon \mathbb{C} \to \mathbb{C}$ defined by $f_c(z) = z^2 + c$. (For instance, $f_{-1}(z) = z^2 - 1$.) Show that $K(f_c)$ is symmetric about the origin by showing that $z \in K(f) \Rightarrow -z \in K(f)$.

Solution: Since $f_c(-z) = (-z)^2 + c = z^2 + c = f(z)$, the iterates of z are bounded if and only if the iterates of $-z$ are bounded.

Problem 5. Go to https://www.marksmath.org/visualization/julia_sets/. There are two copies of $\mathbb{C}$ pictured on that page. Clicking a point on the left side selects a point $c \in \mathbb{C}$, and the number c is displaying in a box underneath. You can choose c without clicking by entering it in this box. The right side then shows the Julia set for $f_c(z) = z^2 + c$.

(a) Enter the point $c = 0$ to see the Julia set for $f_{-1}(z) = z^2$. (You will see the point displayed in the set on the left.)

(b) Enter the point $c = -1$ to see the Julia set for $f_{-1}(z) = z^2 - 1$.

(c) What happens as you click points along the real axis going from 0 to -1?

(d) Hit "Clear" to erase the Julia sets drawn so far. The shape pictured in the right is the Mandelbrot set, M. It is the set of points $c \in \mathbb{C}$ such that the iterates of 0 under the mapping $f_c(z) = z^2 + c$ are bounded, i.e., $0, c, c^2 + c, (c^2 + c)^2 + c, \ldots$ is bounded. What distinguishes Julia sets for $c \in M$ and $c \notin M$?

Problem 6. Show that $K(f_c)$ is symmetric about the real axis.

Problem 7. Prove that for all $c \in \mathbb{C}$, we have $K(f_c) \neq \emptyset$.