PROBLEM 1. Let $a_n = (-1)^n$, and let $a = 0$. Is the following statement true or false? Provide a proof or explicit counterexample.

For all $N \in \mathbb{R}$ there is an $\varepsilon > 0$, such that if $n > N$, then $|a - a_n| < \varepsilon$.

What is the relevance of the above statement to the question of the convergence or divergence of $\{a_n\}$?

SOLUTION: The statement is true. For instance, take $\varepsilon = 10$. Then

$$|a - a_n| = |0 - (-1)^n| = 1 < 10 = \varepsilon$$

for all n. So no matter the choice of N, we have $n > N$ implies that $|a - a_n| < 10$.

The statement, although vaguely similar to the definition of the limit, is irrelevant to the question of convergence.

PROBLEM 2. Let $a_n = 1/n$ for $n \geq 1$, and let $a = 0$. Is the following statement true or false? Provide a proof or explicit counterexample.

For all $\varepsilon > 0$ and $N \in \mathbb{R}$, if $n > N$, then $|a - a_n| < \varepsilon$.

What is the relevance of the above statement to the question of the convergence or divergence of $\{a_n\}$?

SOLUTION: The statement is false. For instance, let $\varepsilon = 1/2$, $N = 0$, and $n = 1$. Then $n > N$, but

$$|a - a_n| = |a - a_1] = |0 - 1/1| = 1 \not< \frac{1}{2} = \varepsilon,$$

i.e., $|a - a_1| \not< \varepsilon$.

Again, although the statement is vaguely similar to the definition of the limit, is irrelevant to the question of convergence.

PROBLEM 3. Find the limit of $\lim_{n\to\infty} \frac{3n^3 + 2n}{6n^3 + 4n + 7}$ and provide an ε-N proof.

SOLUTION: Claim: $\lim_{n\to\infty} \frac{3n^3 + 2n}{6n^3 + 4n + 7} = \frac{1}{2}$.

Proof. Given $\varepsilon > 0$, let $N = \max\left\{1, \frac{7}{12\varepsilon}\right\}$ and suppose $n > N$. Then

$$\left|\frac{1}{2} - \frac{3n^3 + 2n}{6n^3 + 4n + 7}\right| = \frac{(6n^3 + 4n + 7) - 2(3n^3 + 2n)}{2\,(6n^3 + 4n + 7)}$$

$$= \frac{7}{12n^3 + 8n + 14}$$

$$
\begin{aligned}
&< \frac{7}{12n^3} \\
&\le \frac{7}{12n} \\
&< \frac{7}{12N} \\
&= \varepsilon.
\end{aligned}
$$

□

PROBLEM 4. (Challenge, if there is extra time.) Prove that $\lim_{n\to\infty} \frac{1}{n} \neq 1$. (Hint: you need to find an explicit $\varepsilon > 0$ that can't be beat by any $N \in \mathbb{R}$.)

Proof. We have $a = 1$ and $a_n = 1/n$. So

$$|a - a_n| = \left|1 - \frac{1}{n}\right| = \frac{n-1}{n}.$$

The question is whether, given arbitrary $\varepsilon > 0$, can we find N such that $n > N$ implies $|a - a_n| < \varepsilon$? Since $|a - a_n|$ is getting close to 1 as n gets large, the answer is no—not for arbitrary ε. Let $\varepsilon = 1/2$, for instance. Then

$$|a - a_n| \ge \varepsilon \quad \Longleftrightarrow \quad \frac{n-1}{n} \ge \frac{1}{2} \quad \Longleftrightarrow \quad 2(n-1) \ge n \quad \Longleftrightarrow \quad n \ge 2.$$

So no matter what the value of $N \in \mathbb{R}$, there will be an $n > N$ such that $|a - a_n| \not< 1/2$. □