In these problems, let $F = \mathbb{R}$ or $\mathbb{C}$. Your proofs should work simultaneously for both. To find your proofs, it might be easier to draw pictures in the complex plane.

Definition. A subset $U \subseteq F$ is *open* if it contains an open ball about each of its points. This means that for all $u \in U$, there exists $\varepsilon > 0$ such that

$$B(u; \varepsilon) \subseteq U,$$

i.e., if $w \in F$ and $|w - u| < \varepsilon$, then $w \in U$.

Proof template. Let U be the subset of F defined by blah, blah, blah. Then U is open.

Steps in direct proof: (1) Let $u \in U$. (2) What should ε be? (3) Argue that $B(u, \varepsilon) \subseteq U$, i.e., that $|w - u| < \varepsilon$ implies $w \in U$. □

PROBLEM 1. Let $z \in \mathbb{C}$. Prove that $\mathbb{C} \setminus \{z\}$ is open. (Hints: Given a point w in the set, what should ε be? Why is the resulting open ball of radius ε about w contained in the original set? Write this down using complete sentences.)

PROBLEM 2. In any topology, the intersection of a finite number of open sets is open. Let $U_1, \ldots, U_k$ be open subsets of $\mathbb{R}$ or $\mathbb{C}$. Prove that $\cap_{i=1}^{k} U_i$ is open. (Hints: Given a point w in the set, what should ε be? Why is the resulting open ball of radius ε about w contained in the original set? Write this down using complete sentences.)

PROBLEM 3. Let I be any index set, and for each $i \in I$, let U_i be an open subset of F. Is $\cap_{i \in I} U_i$ necessarily open? Give a proof or a (concrete) counterexample.