In these problems, let $F = \mathbb{R}$ or $\mathbb{C}$. Your proofs should work simultaneously for both. To find your proofs, it might be easier to draw pictures in the complex plane.

Definition. A subset $U \subseteq F$ is *open* if it contains an open ball about each of its points. This means that for all $u \in U$, there exists $\varepsilon > 0$ such that

$$B(u; \varepsilon) \subseteq U,$$

i.e., if $w \in F$ and $|w - u| < \varepsilon$, then $w \in U$.

Proof template. Let U be the subset of F defined by blah, blah, blah. Then U is open.

Steps in direct proof: (1) Let $u \in U$. (2) What should ε be? (3) Argue that $B(u, \varepsilon) \subseteq U$, i.e., that $|w - u| < \varepsilon$ implies $w \in U$. □

Problem 1. Let $z \in \mathbb{C}$. Prove that $\mathbb{C} \setminus \{z\}$ is open. (Hints: Given a point w in the set, what should ε be? Why is the resulting open ball of radius ε about w contained in the original set? Write this down using complete sentences.)

Proof. Given $w \in \mathbb{C} \setminus \{z\}$, let $\varepsilon := |w - z|$. We claim that $B(w; \varepsilon) \subseteq \mathbb{C} \setminus \{z\}$. It suffices to show that $z \notin B(w; \varepsilon)$. To see this, note that

$$|z - w| =: \varepsilon \not< \varepsilon.$$ □

Problem 2. In any topology, the intersection of a finite number of open sets is open. Let $U_1, \ldots, U_k$ be open subsets of $\mathbb{R}$ or $\mathbb{C}$. Prove that $\cap_{i=1}^k U_i$ is open. (Hints: Given a point w in the set, what should ε be? Why is the resulting open ball of radius ε about w contained in the original set? Write this down using complete sentences.)

Proof. Let $u \in \cap_{i=1}^k U_i$. Then $u \in U_i$ for $i = 1, \ldots, k$. For each i, since U_i is open, there exists $\varepsilon_i > 0$ such that $B(u; \varepsilon_i) \subseteq U_i$. Define $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_k\}$. Then $\varepsilon > 0$. We claim $B(u; \varepsilon) \subseteq \cap_{i=1}^k U_i$. To see this, let $w \in B(u; \varepsilon)$. For each i, we have

$$|w - u| < \varepsilon \leq \varepsilon_i.$$

Hence, $w \in B(u; \varepsilon_i) \subseteq U_i$ for each i, and so $w \in U_i$. Since $w \in U_i$ for all i, it follows that $w \in \cap_{i=1}^k U_i$. □

Problem 3. Let I be any index set, and for each $i \in I$, let U_i be an open subset of F. Is $\cap_{i \in I} U_i$ necessarily open? Give a proof or a (concrete) counterexample.

Solution: We have just shown that finite intersections of open sets are open. However, an infinite intersection of open sets is not necessarily open. For example, let $\varepsilon_i = 1/i$, and let $U_i = B(0, \varepsilon_i) \subset \mathbb{C}$ for $i = 1, 2, 3, \ldots$ Then $\cap_{i \geq 1} U_i = \{0\} \in \mathbb{C}$. The set $\{0\}$ is not open since every open set containing $\{0\}$ contains nonzero points, too. □