PROBLEM 1. Give an ε-N proof that

$$\lim_{n\to\infty} \frac{\cos(n) + \sqrt{2}i\sin(n)}{n} = 0.$$

(Hint: the triangle inequality is your friend.)

Proof. Given $\varepsilon > 0$, let $N = 3/\varepsilon$. If $n > N$, it follows that

$$\begin{aligned}
\left|0 - \frac{\cos(n) + \sqrt{2}i\sin(n)}{n}\right| &= \left|\frac{\cos(n) + \sqrt{2}i\sin(n)}{n}\right| \\
&= \frac{|\cos(n) + \sqrt{2}i\sin(n)|}{n} \\
&\leq \frac{|\cos(n)| + |\sqrt{2}i\sin(n)|}{n} \\
&= \frac{|\cos(n)| + \sqrt{2}|\sin(n)|}{n} \\
&\leq \frac{1+\sqrt{2}}{n} \\
&\leq \frac{3}{n} \\
&< \frac{3}{N} \\
&= \varepsilon.
\end{aligned}$$

□

PROBLEM 2. Give an ε-N proof that

$$\lim_{n\to\infty} \frac{n}{4n^3 + 2n^2 + 5n + 1} = 0.$$

Proof. Given $\varepsilon > 0$, let $N = \sqrt[3]{\varepsilon}$. If $n > N$, then

$$\left|0 - \frac{1}{4n^3 + 2n^2 + 5n + 1}\right| = \frac{1}{4n^3 + 2n^2 + 5n + 1} < \frac{1}{4n^3} < \frac{1}{n^3} < \frac{1}{N^3} = \varepsilon.$$

□

Problem 3. Give an ε-N proof that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0.$$

Proof. Given $\varepsilon > 0$, let $N = 1/\varepsilon^2$. If $n > N$, then

$$\left|0 - \frac{1}{\sqrt{n+1}+\sqrt{n}}\right| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \varepsilon.$$

□

Problem 4. Does the sequence $\left\{\sqrt{n+1} - \sqrt{n}\right\}$ converge? Proof?

Solution. Claim: $\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n}) = 0$.

Proof. Given $\varepsilon > 0$, let $N = 1/\varepsilon^2$. Then if $n > N$, we have

$$\begin{aligned}
|0 - (\sqrt{n+1} - \sqrt{n})| &= |\sqrt{n+1} - \sqrt{n}| \\
&= \left|\left(\frac{\sqrt{n+1} - \sqrt{n}}{1}\right) \cdot \left(\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right)\right| \\
&= \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} \\
&= \frac{1}{\sqrt{n+1} + \sqrt{n}} \\
&< \frac{1}{2\sqrt{n}} \\
&< \frac{1}{\sqrt{n}} \\
&< \frac{1}{\sqrt{N}} \\
&= \varepsilon.
\end{aligned}$$

□

Problem 5. (Challenge, if you have extra time.)
Does $\{\frac{n!}{n^n}\}$ converge? (Hint: write $n!/n^n$ as a product of n distinct factors, and try to bound it above by a nice function of n.)

Solution: Claim $\displaystyle\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

Proof. Write $n!/n^n$ as the product of n distinct terms:

$$\frac{n!}{n^n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{2}{n} \cdot \frac{1}{n}.$$

Notice that $k/n \leq 1$ for $k = 2, 3, \ldots, n$, so each of the first $n-1$ terms in the above product is bounded above by 1. It follows that

$$0 \leq \frac{n!}{n^n} \leq \frac{1}{n}$$

for $n \geq 1$.

Given $\varepsilon > 0$, let $N = 1/\varepsilon$. If $n > N$, then using what we have just learned,

$$\left|0 - \frac{n!}{n^n}\right| = \frac{n!}{n^n} \leq \frac{1}{n} < \frac{1}{N} = \varepsilon.$$

□