PROBLEM 1. Compute and write in standard form ($a + bi$ with $a, b \in \mathbb{R}$):

(a) $\overline{9 - 6i}$
(b) $|-3 + 2i|$
(c) $(-3 + 2i)^2$
(d) $(1 + i)/(1 - i)$
(e) $\mathrm{Im}((1 + i)/(1 - i))$.

Solution.

(a) $9 + 6i$
(b) $\sqrt{13}$
(c) $5 - 12i$
(d) i
(e) 1.

PROBLEM 2. Let $z = \cos(\theta) + \sin(\theta)i$ for some $\theta \in [0, 2\pi)$.

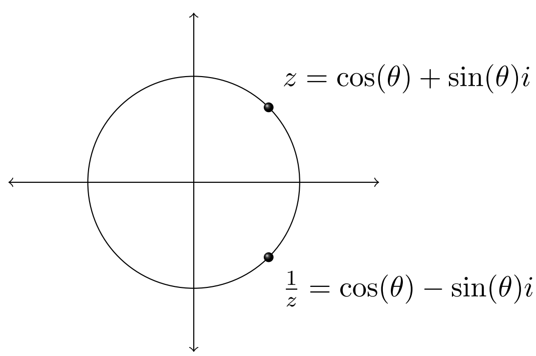
(a) Express $1/z$ in the form $a + bi$ with $a, b \in \mathbb{R}$.
(b) Plot z and $1/z$ for various values of θ. How are z and $1/z$ related geometrically?

Solution.

(a) We have

$$\begin{aligned}\frac{1}{\cos(\theta) + \sin(\theta)i} &= \frac{1}{\cos(\theta) + \sin(\theta)i} \frac{\cos(\theta) - \sin(\theta)i}{\cos(\theta) - \sin(\theta)i} \\ &= \frac{\cos(\theta) - \sin(\theta)i}{\cos^2(\theta) + \sin^2(\theta)} \\ &= \cos(\theta) - \sin(\theta)i.\end{aligned}$$

(b) The multiplicative inverse of $z = \cos(\theta) + \sin(\theta)i$ is obtained by reflecting z across the x-axis:



Problem 3. Let $z = (\sqrt{2}/2, \sqrt{2}/2)$. Compute and plot z^n in the plane for $n \geq 0$. (By definition $z^0 = 1$. Plot $1, z, z^2, z^3, \ldots,$ in turn. A pattern will eventually arise.)

Solution.

$$z^0 = 1 = (1, 0)$$

$$z^1 = (\sqrt{2}/2, \sqrt{2}/2)$$

$$z^2 = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 = i = (0, 1)$$

$$z^3 = z \cdot z^2 = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) i = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$z^4 = z^2 \cdot z^2 = i^2 = -1 = (-1, 0)$$

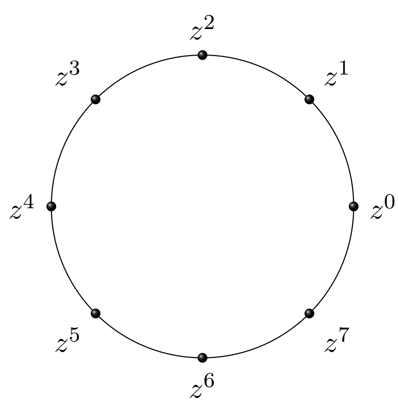
$$z^5 = z \cdot z^4 = -z = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

$$z^6 = z^2 \cdot z^4 = -z^2 = -i = (0, -1)$$

$$z^7 = z^3 \cdot z^4 = -z^3 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

$$z^8 = z^4 \cdot z^4 = -z^4 = -(-1) = 1 = (1, 0).$$

Here is a plot:



The point z is called an *eighth root of unity.* It's eighth power is 1 (and no lower power is 1). Note also that $z^2 = i$. So z is a square root of a square root of -1.

Problem 4. Let $z \in \mathbb{C}$. Prove that $|z| \geq |\operatorname{Im}(z)|$.

Proof. Say $z = a + bi$. We have
$$|z| = \sqrt{a^2 + b^2} \geq \sqrt{b^2} = |b| = |\operatorname{Im}(z)|.$$
$\square$