PROBLEM 1. (Square roots of -1.) For $n \in \{2, 3, 4, 5, 6, 10\}$, find all $x \in \mathbb{Z}/n\mathbb{Z}$ such that $x^2 = -1$.

Solution. In $\mathbb{Z}/2\mathbb{Z}$, we have $1^2 = 1 = -1$. In $\mathbb{Z}/5\mathbb{Z}$, we have $2^2 = 3^2 = 4 = -1$. In $\mathbb{Z}/10/\mathbb{Z}$, we have $3^2 = 7^2 = -1$. For the other values of n, there are no elements x such that $x^2 = -1$.

PROBLEM 2. Prove that $\mathbb{C}$ satisfies the additive associativity axiom. (Use the definition of $\mathbb{C}$, taking one step at a time, justifying each step. You will need to use the definition of addition for $\mathbb{C}$ and associativity of addition for $\mathbb{R}$.)

Proof. Let $(a, b), (a', b'), (a'', b'') \in \mathbb{C}$. Then

$$\begin{aligned}
\big((a, b) + (a', b')\big) + (a'', b'') &= (a + a', b + b') + (a'', b'') && \text{(def. of addition for } \mathbb{C})\\
&= ((a + a') + a'', (b + b') + b'') && \text{(def. of addition for } \mathbb{C})\\
&= (a + (a' + a''), b + (b' + b'')) && \text{(assoc. of addition for } \mathbb{R})\\
&= (a, b) + (a' + a'', b' + b'') && \text{(def. of addition for } \mathbb{C})\\
&= (a, b) + \big((a', b') + (a'', b'')\big) && \text{(def. of addition for } \mathbb{C})
\end{aligned}$$

□

PROBLEM 3. Consider the set $\mathbb{R}^2$ with addition and multiplication defined by

$$(a, b) + (c, d) = (a + c, b + d) \qquad \text{and} \qquad (a, b) \cdot (c, d) = (ac, bd),$$

respectively. Indicate which field axioms fail, giving a concrete counter-example in each case. What are the additive and multiplicative identities? (To save time, you may assume the fact that associativity of addition and multiplication hold.)

*Solution.*The existence of multiplicative inverses fails. First, note that $(0, 0)$ is the additive identity and $(1, 1)$ is the multiplicative identity. The element $(1, 0)$ is nonzero, because $(1, 0) \neq (0, 0)$, yet $(1, 0)$ has no multiplicative inverse: if $(a, b) \in \mathbb{R}^2$ satisfies

$$(1, 1) = (1, 0) \cdot (a, b) = (a, 0),$$

then we must have $1 = 0$ in $\mathbb{R}$, which is absurd.

PROBLEM 4. Compute the following, expressing your result in the form $a + bi$ for $a, b \in \mathbb{R}$.

(a) $(3 + 2i)(-2 + 3i) + (1 + 4i)$

(b) $(2 + 3i)^{-1}$

(c) $\frac{1+4i}{2+i}$.

Solution.

(a)

$$(3 + 2i)(-2 + 3i) + (1 + 4i) = (-12 + 5i) + (1 + 4i) = -11 + 9i.$$

(b)

$$\frac{1}{2+3i} = \frac{1}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{2-3i}{2^2+3^2} = \frac{2}{13} - \frac{3}{13}i.$$

(c)

$$\frac{1+4i}{2+i} = \frac{1+4i}{2+i} \cdot \frac{2-i}{2-i} = \frac{6+7i}{2^2+1^2} = \frac{6}{5} + \frac{7}{5}i.$$