Let $n \geq 1$. A solution $z \in \mathbb{C}$ to the equation $z^n = 1$ is called an *n-th root of unity*. Our goal is to find all of them.

PROBLEM 1. For each $n \in \{2, 3, 4\}$, (i) find all $z \in \mathbb{C}$ such that $z^n = 1$ using algebra, (ii) find the polar form for each solution, and (iii) draw the solutions in the complex plane. (The case for each n should be on a separate page, finding the solutions to $z^n - 1 = 0$.)

(a) $n = 2$

(b) $n = 3$ (Hint: $z - 1$ is a factor of $z^3 - 1$. So $z^3 - 1 = (z-1)(az^2 + bz + c)$ for some $a, b, c \in \mathbb{C}$. Long division could help.)

(c) $n = 4$ (Hint: factor!)

PROBLEM 2. If $z^n = 1$ for some $n \geq 1$, prove that z lies on the unit circle in the complex plane.

PROBLEM 3. Use the intuition you have developed so far to find the polar forms for all n-th roots of unity.

PROBLEM 4. (If there is extra time.) Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, and consider the mapping

$$\begin{aligned} f\colon \mathbb{C}^* &\to \mathbb{C} \\ z &\mapsto \frac{1}{\bar{z}}. \end{aligned}$$

(a) How does this mapping transform the modulus and the argument of each point in $\mathbb{C}^*$.

(b) Think of f as a geometric transformation. It takes the punctured plane $\mathbb{C}^*$, warps it somehow, and sends it back to the plane $\mathbb{C}^*$. Describe the process in geometric terms.

(Hint: writing z in polar form will help.)

(OVER)

Problem 5. (If there is extra time.) The graph of a function $f\colon A \to B$ is the set $\{(x, f(x)) \in A \times B\}$. Thus, if $A = B = \mathbb{R}$, we can draw the graph in $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. What if f is a complex function, i.e., if $A = B = \mathbb{C}$? Then the graph sits in $\mathbb{C} \times \mathbb{C} = \mathbb{R}^2 \times \mathbb{R}^2 \simeq \mathbb{R}^4 = \{(a, b, c, d) : a, b, c, d \in \mathbb{R}\}$. Since the graph of a function sits in four-dimensional space, we need a different strategy to picture these functions. Here is one way. Suppose $f(z) = (a, b) \in \mathbb{C} = \mathbb{R}^2$. We encode $a \in \mathbb{R}$ as color, and $b \in \mathbb{R}$ as brightness. (To do that, we need to choose reasonable functions $a \mapsto$ color and $b \mapsto$ brightness.) To picture the graph of f, we go to each point $z \in \mathbb{C} = \mathbb{R}^2$, and color that point with its corresponding color and brightness.

Go to https://sagecell.sagemath.org/ and type in the following to see the graph of $f(z) = z^3 - 1$:

```
p = complex_plot(lambda z: z^3-1, (-2, 2), (-2, 2))
p.show(aspect_ratio=1)
```

(a) Try graphing the functions $z^n = 1$ for various values of n. How do you connect these pictures to the roots of unity?

(b) Try graphing some other functions, e.g, `z^2+2*z+1` (note the $*$ for multiplication). A polynomial of degree 3? What about trig functions? The mysterious Riemann zeta function `zeta(z)`? For the zeta function, you might want to zoom out by replacing `(-2,2), (-2,2)` by `(-20,20), (-20,20)`.