Let $n \geq 1$. A solution $z \in \mathbb{C}$ to the equation $z^n = 1$ is called an *$n$-th root of unity*. Our goal is to find all of them.

PROBLEM 1. For each $n \in \{2, 3, 4\}$, (i) find all $z \in \mathbb{C}$ such that $z^n = 1$ using algebra, (ii) find the polar form for each solution, and (iii) draw the solutions in the complex plane. (The case for each $n$ should be on a separate page, finding the solutions to $z^n - 1 = 0$.)

(a) $n = 2$

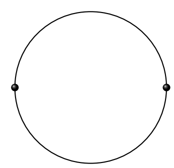
(b) $n = 3$ (Hint: $z-1$ is a factor of $z^3-1$. So $z^3-1 = (z-1)(az^2+bz+c)$ for some $a, b, c \in \mathbb{C}$. Long division could help.)

(c) $n = 4$ (Hint: factor!)

*Solution.*

(a) $z^2 - 1 = (z-1)(z+1) = 0$ has solutions $z = \pm 1$. In polar form:

$$1 = \cos(0) + \sin(0)i, \quad -1 = \cos(\pi) + \sin(\pi)i$$

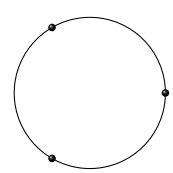


(b) $z^3 - 1 = (z-1)(z^2+z+1) = 0$ when $z = 1$ or when $z^2 + z + 1 = 0$. Use the quadratic equation to find the solutions it the latter:

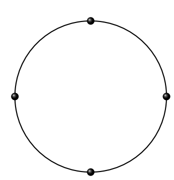
$$z = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\,i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\,i.$$

Polar forms

$$\begin{aligned} 1 &= \cos(0) + \sin(0)i \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i &= \cos(2\pi/3) + \sin(2\pi/3) \\ -\frac{1}{2} - \frac{\sqrt{3}}{2}\,i &= \cos(8\pi/6) + \sin(8\pi/6). \end{aligned}$$



(c) $z^4 - 1 = (z^2+1)(z^2-1) = 0$ if and only if $z^2 = -1$ or $z^2 = 1$. So the solutions are $\pm i$ and $z$:



Problem 2. If $z^n = 1$ for some $n \geq 1$, prove that $z$ lies on the unit circle in the complex plane.

*Proof.* We have

$$z^n = 1 \quad \Rightarrow \quad |z^n| = |1| \quad \Rightarrow \quad |z|^n = 1 \quad \Rightarrow \quad |z| = 1.$$

□

Problem 3. Use the intuition you have developed so far to find the polar forms for all $n$-th roots of unity.

*Solution.* The $n$-th roots of unity are

$$\cos\left(\frac{2k\pi}{n}\right) + \sin\left(\frac{2k\pi}{n}\right) i$$

for $k = 0, 1, \ldots, n-1$.

Problem 4. (If there is extra time.) Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, and consider the mapping

$$\begin{aligned} f\colon \mathbb{C}^* &\to \mathbb{C} \\ z &\mapsto \frac{1}{\bar{z}}. \end{aligned}$$

(a) How does this mapping transform the modulus and the argument of each point in $\mathbb{C}^*$.

(b) Think of $f$ as a geometric transformation. It takes the punctured plane $\mathbb{C}^*$, warps it somehow, and sends it back to the plane $\mathbb{C}^*$. Describe the process in geometric terms.

(Hint: writing $z$ in polar form will help.)

*Solution.*

(a) Write $z \in \mathbb{C}^*$ in polar form as $z = |z|(\cos(\theta) + i\sin(\theta)$. Then

$$\begin{aligned} \frac{1}{\bar{z}} &= \frac{1}{|z|(\cos(\theta) - \sin(\theta))} \\ &= \frac{1}{|z|}\frac{1}{\cos(\theta) - i\sin(\theta)} \\ &= \frac{1}{|z|}\frac{1}{\cos(\theta) - i\sin(\theta)}\frac{\cos(\theta) + i\sin(\theta)}{\cos(\theta) + i\sin(\theta)} \\ &= \frac{1}{|z|}(\cos(\theta) + i\sin(\theta)). \end{aligned}$$

Thus,

$$|f(z)| = \frac{1}{|z|} \quad \text{and} \quad \arg(f(z)) = \arg(z).$$

For instance, the image of a circle of radius $r$ centered at the origin is again a circle, but of radius $1/r$. Very small circles about the origin are sent to very large circles, and vice versa. In this way, the plane is turned inside out.

Problem 5. (If there is extra time.) The graph of a function $f\colon A \to B$ is the set $\{(x, f(x)) \in A \times B\}$. Thus, if $A = B = \mathbb{R}$, we can draw the graph in $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. What if $f$ is a complex function, i.e., if $A = B = \mathbb{C}$? Then the graph sits in $\mathbb{C} \times \mathbb{C} = \mathbb{R}^2 \times \mathbb{R}^2 \simeq \mathbb{R}^4 = \{(a, b, c, d) : a, b, c, d \in \mathbb{R}\}$. Since the graph of a function sits in four-dimensional space, we need a different strategy to picture these functions. Here is one way. Suppose $f(z) = (a, b) \in \mathbb{C} = \mathbb{R}^2$. We encode $a \in \mathbb{R}$ as color, and $b \in \mathbb{R}$ as brightness. (To do that, we need to choose reasonable functions $a \mapsto$ color and $b \mapsto$ brightness.) To picture the graph of $f$, we go to each point $z \in \mathbb{C} = \mathbb{R}^2$, and color that point with its corresponding color and brightness.

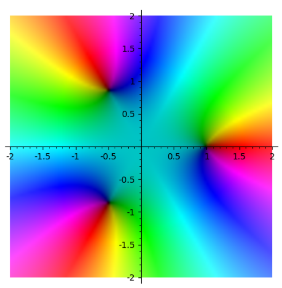
Go to https://sagecell.sagemath.org/ and type in the following to see the graph of $f(z) = z^3 - 1$:

```
p = complex_plot(lambda z: z^3-1, (-2, 2), (-2, 2))
p.show(aspect_ratio=1)
```

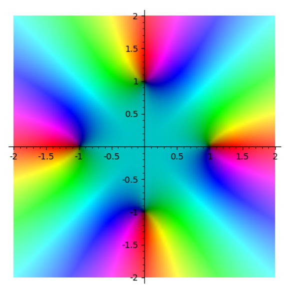
(a) Try graphing the functions $z^n = 1$ for various values of $n$. How do you connect these pictures to the roots of unity?

(b) Try graphing some other functions, e.g, `z^2+2*z+1` (note the $*$ for multiplication). A polynomial of degree 3? What about trig functions? The mysterious Riemann zeta function `zeta(z)`? For the zeta function, you might want to zoom out by replacing `(-2,2), (-2,2)` by `(-20,20), (-20,20)`.

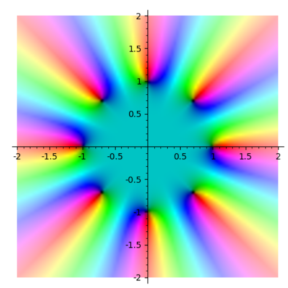
*Solution.*



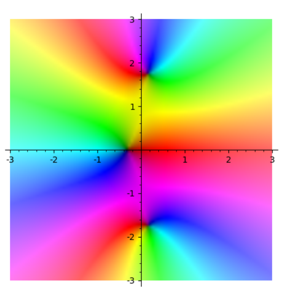
$z^3 - 1$



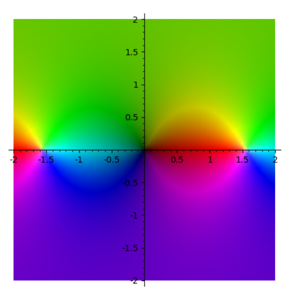
$z^4 - 1$



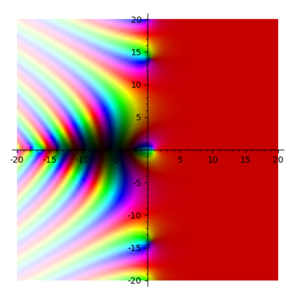
$z^8 - 1$



$z^3 + 2z + z + 1$



$\tan(z)$



$\zeta(z)$