Recall the interval notion for subsets of the reals:

$(a,b) := \{x \in \mathbb{R} : a < x < b\}\,,$ $[a,b) := \{x \in \mathbb{R} : a \leq x < b\}\,,$ $(a,b] := \{x \in \mathbb{R} : a < x \leq b\}\,,$

$[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}\,,$ $(-\infty,b) := \{x \in \mathbb{R} : x < b\}\,,$ $(-\infty,b] := \{x \in \mathbb{R} : x \leq b\}\,,$

$(a,\infty) := \{x \in \mathbb{R} : x > a\}\,,$ $[a,\infty) := \{x \in \mathbb{R} : x \geq a\}\,,$ $(\infty,\infty) := \mathbb{R}.$

Recall the following definitions pertaining to a subset S of an ordered field F:

» $B \in F$ is an *upper bound* for S if $s \leq B$ for all $s \in S$,

» $b \in F$ is an *lower bound* for S if $b \leq s$ for all $s \in S$,

» S is *bounded* if it has both an upper bound and a lower bound.

» $B \in F$ is a *supremum* for S if it is a least upper bound. This means that B is an upper bound and if B' is any upper bound, then $B \leq B'$. If B exists, then we write $B = \sup(S)$ or $B = \operatorname{lub}(S)$.

» $b \in F$ is a *infimum* for S if it is a greatest lower bound. This means that b is a lower bound and if b' is any lower bound, then $b' \leq b$. If b exists, then we write $b = \inf(S)$ or $b = \operatorname{glb}(S)$.

» If S has a supremum B and $B \in S$, then we call B the *maximum* or *maximal element* of S and write $\max(S) = B$.

» If S has in infimum b and $b \in S$, then we call b the *minimum* of *minimal element* of S and write $\min(S) = b$.

Finally, recall that $\mathbb{R}$ satisfies the *completeness axiom*: every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum.

Problem 1. Let $S = [0, 1) \subset \mathbb{R}$.

(a) Give three upper bounds and three lower bounds for S.

(b) Is S bounded? (Appeal to the definition of *bounded* here.)

(c) Does S have a supremum? If so, what is it? Same question for infimum.

(d) Does S have a maximum? a minimum?

Solution.

(a) For example, $1, 7$ and 10^6 are upper bounds and $0, -3$, and -23 are lower bounds.

(b) Yes, since S is bounded above and bounded below.

(c) The supremum of S is 1 and the infimum is 0.

(d) Since $\sup(S) = 1 \notin S$, the set S has no maximum. On the other hand, $\inf(S) = 0 \in S$. Thus, $\min(S) = 0$.

Problem 2. These questions concern the ordered field of rational numbers $\mathbb{Q}$, not the field $\mathbb{R}$. Let $S = (0, \pi) \cap \mathbb{Q}$, a subset of $\mathbb{Q}$.

(a) Is S bounded?

(b) Does S have a supremum?

Solution.

(a) Yes. For instance, $4 \in \mathbb{Q}$ is an upper bound and $0 \in \mathbb{Q}$ is a lower bound.

(b) Since $\pi \notin \mathbb{Q}$ the set S has no supremum (in $\mathbb{Q}$).

Problem 3. Here we're are considering subsets of $\mathbb{R}$. Fill in the following table, using "DNE" if the quantity does not exist:

	sup	max	inf	min
$[-1, 2)$				
$(-1, 2) \cup [3, 4]$				
$[3, \infty) \cup [3, 4]$				
$\mathbb{Z}_{\geq 0}$				
$\{-7, \sqrt{2}, 8, 23\}$				
$\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$				
$\cap_{n=1}^{\infty}(1 - 1/n, 1 + 1/n)$				
$\cup_{n=1}^{\infty}(1 - 1/n, 1 + 1/n)$				

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Solution.

	sup	max	inf	min
$[-1, 2)$	2	DNE	-1	-1
$(-1, 2) \cup [3, 4]$	4	4	-1	DNE
$[3, \infty) \cup [3, 4]$	DNE	DNE	-1	DNE
$\mathbb{Z}_{\geq 0}$	DNE	DNE	0	0
$\{-7, \sqrt{2}, 8, 23\}$	23	23	-7	-7
$\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$	1	DNE	$\frac{1}{2}$	$\frac{1}{2}$
$\cap_{n=1}^{\infty}(1 - 1/n, 1 + 1/n)$	1	1	1	1
$\cup_{n=1}^{\infty}(1 - 1/n, 1 + 1/n)$	2	DNE	0	DNE

Note that $\cap_{n=1}^{\infty}(1 - 1/n, 1 + 1/n) = \{1\}$, and $\cup_{n=1}^{\infty}(1 - 1/n, 1 + 1/n) = (0, 2)$.

PROBLEM 4. Mark each of the following statements as true or false. In each case, give a brief explanation if it is true or a specific counterexample if it is false. Throughout, S denotes a nonempty subset of $\mathbb{R}$.

(a) If S has an upper bound, then S has a least upper bound.
(b) If S is bounded, then S has a maximum and a minimum.
(c) If $S \subseteq \mathbb{Q}$ and S is bounded, then $\sup S \in \mathbb{Q}$.
(d) If $m = \inf S$ and $m' < m$, then m' is a lower bound of S.

Solution.

(a) True by the completeness axiom.
(b) False. A counterexample is $(0, 1)$.
(c) False. A counterexample is given in an earlier problem: $(0, \pi) \cap \mathbb{Q}$.
(d) True. If $s \in S$, then it follows from the definition of the infimum that $m < s$. If $m' < m$, then by transitivity of $<$, we have $m' < s$, too.