PROBLEM 1. Let F be an ordered field, and let $w, x, y, z \in F$. Use the order axioms to prove that if $w < x$ and $y < z$, then $w+y < x+z$. In other words, we can "add inequalities".

Proof. By additive translation, $w < x$ implies $w+y < x+y$. Similarly, $y < z$ implies $x+y < x+z$. Our result then follows by transitivity:

$$w + y < x + y \quad \text{and} \quad x + y < x + z \quad \Longrightarrow \quad w + y < x + z.$$

□

PROBLEM 2. Let F be an ordered field, and let $x \in F$ with $x > 0$. Since F is a field, x has a multiplicative inverse, $1/x$. Prove that $1/x > 0$. [Hint: break the possibilities for $1/x$ into cases using trichotomy, and rule out two of those cases.]

Proof. By trichotomy, exactly one of the following holds:

$$\frac{1}{x} = 0, \quad \frac{1}{x} < 0, \quad \text{or} \quad \frac{1}{x} > 0.$$

We will prove the result by ruling out the first two possibilities. First, using the definition of $1/x$ and the fact that $x \cdot 0 = 0$, as shown previously, note that

$$\frac{1}{x} = 0 \quad \Rightarrow \quad x \cdot \frac{1}{x} = x \cdot 0 \quad \Rightarrow \quad 1 = 0.$$

This cannot be, since $1 \neq 0$ in any field (as dictated explicitly in the definition of a field). Thus, $1/x \neq 0$.

Next, using multiplicative translation and the fact that $x > 0$,

$$\frac{1}{x} < 0 \quad \Rightarrow \quad x \cdot \frac{1}{x} < 0 \quad \Rightarrow \quad 1 < 0.$$

However, we have seen that in any field, $1 > 0$.

By process of elimination, we have $1/x > 0$. □

PROBLEM 3. Can the field $\mathbb{Z}/5\mathbb{Z}$ be ordered? In other words, does there exist a relation on $\mathbb{Z}/5\mathbb{Z}$ satisfying the order axioms? [Hint: from the lecture notes, we know that for any nonzero element x of an ordered field, we have $x^2 > 0$. In particular, this means that $1 > 0$ since $1 = 1^2$. Start with $1 > 0$.]

Solution. For convenience, denote the elements of $\mathbb{Z}/5\mathbb{Z}$ by $0, 1, 2, 3, 4$, dropping the usual square brackets.

For the sake of contradiction, suppose that $\mathbb{Z}/5\mathbb{Z}$ could be ordered. We would then have $0 < 1$, and by repeated use of additive translation,

$$0 < 1 \Rightarrow 1 < 2 \Rightarrow 2 < 3 \Rightarrow 3 < 4 \Rightarrow 4 < 0.$$

By repeated application of transitivity, it would then follow that $0 < 0$, which violates trichotomy.