Recall the following definitions pertaining to a subset S of an ordered field F:

- » $B \in F$ is an *upper bound* for S if $s \leq B$ for all $s \in S$,
- » $b \in F$ is an *lower bound* for S if $b \leq s$ for all $s \in S$,
- » S is *bounded* if it has both an upper bound and a lower bound.
- » $B \in F$ is a *supremum* for S if it is a least upper bound. This means that B is an upper bound and if B' is any upper bound, then $B \leq B'$. If B exists, then we write $B = \sup(S)$ or $B = \text{lub}(S)$.
- » $b \in F$ is a *infimum* for S if it is a greatest lower bound. This means that b is a lower bound and if b' is any lower bound, then $b' \leq b$. If b exists, then we write $b = \inf(S)$ or $b = \text{glb}(S)$.
- » If S has a supremum B and $B \in S$, then we call B the *maximum* or *maximal element* of S and write $\max(S) = B$.
- » If S has in infimum b and $b \in S$, then we call b the *minimum* of *minimal element* of S and write $\min(S) = b$.

Recall that $\mathbb{R}$ satisfies the *completeness axiom*: every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum.

Problem 1. Here were are considering subsets of $\mathbb{R}$. Fill in the following table, using "DNE" if the quantity does not exist:

	sup	max	inf	min
$\left\{\frac{1}{2n} : n \in \mathbb{N}_{>0}\right\}$				
$\left\{(-1)^n\left(1+\frac{1}{n}\right) : n \in \mathbb{N}_{>0}\right\}$				

Problem 2. Mark each of the following statements as true or false. In each case, give a brief explanation if it is true or a specific counterexample if it is false. Throughout, S denotes a nonempty subset of $\mathbb{R}$.

(a) If $B = \sup S$ and $B' < B$, then B' is an upper bound of S.

(b) If $B = \sup S$ and $B < B'$, then B' is an upper bound of S.

(c) $\emptyset$ is bounded.

(d) $\sup \emptyset$ and $\inf \emptyset$ do not exist.

Problem 3. Your answer to the last two parts of the previous problem shows that $\mathbb{R}$ has a subset that is bounded above but that has no supremum. Why doesn't that contradict the fact that $\mathbb{R}$ is complete.

Problem 4. Suppose that $\emptyset \neq X \subseteq S \subset \mathbb{R}$ and S has an supremum. Prove that

(a) $\sup(S)$ is an upper bound for X,

(b) $\sup X$ exists, and

(c) $\sup X \leq \sup S$.

(For part (a), use the template from the reading.)

Problem 5. Let S be a subset of an ordered field F.

Recall the definition of the supremum: $B \in F$ is a *supremum* for S if it is a least upper bound. This means that B is an upper bound and if B' is any upper bound, then $B \leq B'$.

Use this definition to show that if u and v are both suprema of S, then $u = v$.