Recall the following definitions pertaining to a subset S of an ordered field F:

- » $B \in F$ is an *upper bound* for S if $s \leq B$ for all $s \in S$,
- » $b \in F$ is an *lower bound* for S if $b \leq s$ for all $s \in S$,
- » S is *bounded* if it has both an upper bound and a lower bound.
- » $B \in F$ is a *supremum* for S if it is a least upper bound. This means that B is an upper bound and if B' is any upper bound, then $B \leq B'$. If B exists, then we write $B = \sup(S)$ or $B = \operatorname{lub}(S)$.
- » $b \in F$ is a *infimum* for S if it is a greatest lower bound. This means that b is a lower bound and if b' is any lower bound, then $b' \leq b$. If b exists, then we write $b = \inf(S)$ or $b = \operatorname{glb}(S)$.
- » If S has a supremum B and $B \in S$, then we call B the *maximum* or *maximal element* of S and write $\max(S) = B$.
- » If S has in infimum b and $b \in S$, then we call b the *minimum* of *minimal element* of S and write $\min(S) = b$.

Recall that $\mathbb{R}$ satisfies the *completeness axiom*: every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum.

PROBLEM 1. Here were are considering subsets of $\mathbb{R}$. Fill in the following table, using "DNE" if the quantity does not exist:

	sup	max	inf	min
$\left\{\frac{1}{2n} : n \in \mathbb{N}_{>0}\right\}$				
$\left\{(-1)^n \left(1+\frac{1}{n}\right) : n \in \mathbb{N}_{>0}\right\}$				

.

Solution.

	sup	max	inf	min
$\left\{\frac{1}{2n} : n \in \mathbb{N}_{>0}\right\}$	$\frac{1}{2}$	$\frac{1}{2}$	0	DNE
$\left\{(-1)^n \left(1+\frac{1}{n}\right) : n \in \mathbb{N}_{>0}\right\}$	$\frac{3}{2}$	$\frac{3}{2}$	-2	-2

.

PROBLEM 2. Mark each of the following statements as true or false. In each case, give a brief explanation if it is true or a specific counterexample if it is false. Throughout, S denotes a nonempty subset of $\mathbb{R}$.

(a) If $B = \sup S$ and $B' < B$, then B' is an upper bound of S.

(b) If $B = \sup S$ and $B < B'$, then B' is an upper bound of S.

(c) $\emptyset$ is bounded.

(d) $\sup \emptyset$ and $\inf \emptyset$ do not exist.

Solution.

(a) False. A counter example is given by $S = (0, 1)$, $B = 1$ and $B' = 1/2$.

(b) True. Suppose $B < B'$. To see B' is an upper bound, let $s \in S$. By definition of the supremum, $s < B$. Then, by transitivity of $<$ it follows that $s < B'$.

(c) Yes. Every real number is both an upper bound and a lower bound for $\emptyset$. For instance, 3 is an upper bound since it is true that $3 > x$ for all $x \in \emptyset$. That's because there there exists no element x in $\emptyset$. Similar reasoning shows that 3 is also a lower bound.

(d) Since every real number is an upper bound for $\emptyset$, it follows that $\emptyset$ has no least upper bound, i.e., it has no supremum. A similar argument shows that $\emptyset$ does not have an infimum.

PROBLEM 3. Your answer to the last two parts of the previous problem shows that $\mathbb{R}$ has a subset that is bounded above but that has no supremum. Why doesn't that contradict the fact that $\mathbb{R}$ is complete.

Solution. The completeness axiom requires that every *nonempty* subset of $\mathbb{R}$ that is bounded above have a supremum.

Problem 4. Suppose that $\emptyset \neq X \subseteq S \subset \mathbb{R}$ and S has an supremum. Prove that

(a) $\sup X$ exists, and

(b) $\sup X \leq \sup S$.

(Hint for part (a): By completeness, you just need to show what about X? What could possibly be an upper bound for X? Hint for part (b): why do you just need to show that $\sup S$ is an upper bound for X?)

Proof.

(a) We first show that X is bounded above by $\sup(S)$. Let $x \in X$. Then, since $X \subseteq S$, we have $x \in S$, and hence $x \leq \sup(S)$. Thus, X is bounded above. Since $X \neq \emptyset$, it follows that from completeness of $\mathbb{R}$ that $\sup(X)$ exists.

(b) We have just shown that $\sup(S)$ is an upper bound for X. It follows from the definition of the supremum of X that $\sup(X) \leq \sup(S)$. (The idea is that $\sup(S)$ is *an* upper bound for X, and $\sup(X)$ is the *least* upper bound for X.)

$\square$

Problem 5. Let S be a subset of an ordered field F.

Recall the definition of the supremum: $B \in F$ is a *supremum* for S if it is a least upper bound. This means that B is an upper bound and if B' is any upper bound, then $B \leq B'$.

Use this definition to show that if u and v are both suprema of S, then $u = v$.

Proof. Suppose u and v are suprema of S. Then since u is *an* upper bound and v is a *least* upper bound, it follows that $v \leq u$. Similarly, since v is *an* upper bound, and u is a *least* upper bound, it follows that $u \leq v$.

Since $v \leq u$ and $u \leq v$, the trichotomy axiom for ordered fields implies that $u = v$. $\square$