Consider the equivalence relation on sets defined by $A \sim B$ if there exists a bijection from A to B. We say two sets have the *same cardinality* if they are equivalent under this equivalence relation, and we write $|A| = |B|$. Any set with the same cardinality as $\mathbb{N}$ is *countably infinite*. For a set A to be countably infinite means that its elements can be listed in an unending line, $a_0, a_1, a_2, \ldots$. (The resulting bijection $\mathbb{N} \to A$ sends n to a_n.) Last time, we showed that the rational numbers are countably infinite.

Problem 1. (Cantor's diagonal argument, 1891) It turns out that the real numbers are not countable, i.e., they cannot be put into bijection with the natural numbers. Here, we will give the slightly easier argument that the subset of the real decimals containing only 0s and 1s is not countable. Define *binary decimals* to be the real numbers of the form $0.a_1a_2a_3\ldots$ where each $a_i \in \{0, 1\}$. A binary decimal would look like $0.0110001010011\ldots$ For sake of contradiction, suppose you could list the binary decimals. Your list would then look something like this (leaving off the initial "0."):

n													
0	1	1	0	0	0	1	0	1	0	0	1	1	...
1	0	0	0	1	0	1	0	0	1	0	0	0	...
2	1	0	0	0	0	0	1	0	0	0	0	1	...
3	0	1	0	0	1	1	0	1	1	1	1	0	...
4	1	0	1	1	1	0	0	0	1	1	0	1	...
5	0	1	1	1	1	1	0	0	0	1	1	1	...
6	1	1	1	1	1	0	0	0	0	0	0	1	...
7	1	0	1	0	0	0	0	0	0	0	0	1	...
8	1	1	1	0	0	0	1	0	0	0	0	0	...
9	0	1	1	0	1	0	0	1	0	1	1	0	...
10	0	0	1	0	0	1	1	0	1	0	1	0	...
11	0	1	0	1	0	1	0	0	0	0	0	1	...
⋮							⋮						

We will show that your list is not complete. Read off the diagonal from the above table: $0.100011000111\ldots$. Except for the initial "0.", swap the 0s and 1s in this number: $0.011100111000\ldots$ Why isn't this number in the list? Next, place this number at the beginning of your list. Do you now have a complete list of the binary decimals?

Solution. It differs from the zero-th number in the list in its first decimal, from the first number in its second decimal, and so on. If we place the newly formed number at the beginning of the list, we can perform the same procedure, going down the diagonal, to produce a binary decimal that is not in this newly formed list. No matter what linear list of binary decimals we create, it will not contain all of the binary decimals.

Problem 2. If A and B are sets, we write $|A| < |B|$ if there exists an injection $A \to B$ but there exists no bijection $A \to B$. Why is it the case that $|\mathbb{N}| < |\mathbb{R}|$? In this way, there are at least two "sizes" for infinite sets.

Solution. There is a natural injection $f \colon \mathbb{N} \to \mathbb{R}$ given by $f(n) = n$, and the previous problem shows there is no bijection.

PROBLEM 3. Let A be a set and let $\mathcal{P}(A)$ be the set of all subsets of A. In this problem, we show that $|A| < |\mathcal{P}(A)|$. Thus, for instance, we see that

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \cdots .$$

(a) If $A = \{1, 2, 3\}$, find $\mathcal{P}(A)$.

(b) Describe an injection $A \to \mathcal{P}(A)$.

(c) We now show that there is no surjection $A \to \mathcal{P}(A)$. Let $f \colon A \to \mathcal{P}(A)$ be any function. Define

$$B = \{a \in A \colon a \notin f(a)\} .$$

We would like to show that B is not in the image of f, i.e., there is no $a \in A$ such that $f(a) = B$. For sake of contradiction, suppose there is an $a \in A$ such that $f(a) = B$. Then either $a \in B$ or $a \notin B$. Is $a \in B$? Is $a \notin B$?

Solution.

(a) We have

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\} .$$

(b) There are lots of them, but here is a natural one:

$$\begin{aligned} A &\mapsto \mathcal{P}(A) \\ a &\to \{a\} . \end{aligned}$$

(c) If $a \in B$, then since $B = f(a)$, we have $a \in f(a)$, which means $a \notin B$. So that cannot be. On the other hand, if $a \notin B$, then since $B = f(a)$, we have $a \notin B$, which means that $a \in B$. So that cannot be, either. It follows that there cannot be an a such that $f(a) = B$, and therefore, there is no surjection $A \to \mathcal{P}(A)$.