If A and B are sets, say $A \sim B$ if there exists a bijection $A \to B$. All of the problems below refer to this relation

PROBLEM 1. Prove that $\sim$ is an equivalence relation.

Proof. Let A, B, C be sets.

Reflexivity. We have $A \sim A$ since the identity mapping $\mathrm{id}_A \colon A \to A$ defined by $\mathrm{id}_A(a) = a$ for all $a \in A$ is a bijection.

Symmetry. Suppose $A \sim B$. Then there exists a bijection $f \colon A \to B$. Since f is bijective, it has an inverse $f^{-1} \colon B \to A$, and that inverse is a bijection. Hence, $B \sim A$.

Transitivity. Suppose $A \sim B$ and $B \sim C$. Then there are bijections $f \colon A \to B$ and $g \colon B \to C$. Since the composition of bijections is a bijection, $g \circ f \colon A \to C$ is a bijection. Hence, $A \sim C$. $\square$

PROBLEM 2. Let A, B be sets. If $A \sim B$, we say that A and B have the *same cardinality* and write $|A| = |B|$.

(a) If A is a finite set, describe all of the sets B in the equivalence class for A.

(b) Let $2\mathbb{Z}$ denote the set of even integers. Prove that $\mathbb{Z}$ and $2\mathbb{Z}$ have the same cardinality.

Solution.

(a) The equivalence class for A consist of all sets that have the same number of elements as A.

(b) The mapping

$$\begin{aligned} \mathbb{Z} &\to 2\mathbb{Z} \\ n &\to 2n \end{aligned}$$

is a bijection.

$\square$

PROBLEM 3. A set having the same cardinality as $\mathbb{N}$ is said to be *countably infinite*. To say that a set X is countably infinite means the its elements may be listed in a line:

$$x_0, x_1, x_2, \ldots$$

Given this list, we get a bijection $f \colon \mathbb{N} \to X$ by letting $f(n) := x_n$. Conversely, given a bijection $f \colon \mathbb{N} \to X$, we create the list as

$$f(0), f(1), f(2), \ldots$$

Prove that $\mathbb{Z}$ is countably infinite.

Solution. Here is a list of the elements of $\mathbb{Z}$:

$$0, -1, 1, -2, 2, -3, 3, \ldots$$

$\square$

PROBLEM 4. Is $\mathbb{Q}$, the set of rational numbers, countably infinite? If so, describe your list of the rationals. Please consider this question for a while before proceeding to the next problem (which provides a solution). If you have prior knowledge regarding this question, please don't give away the solution to your fellow group members.

PROBLEM 5. Fill in the following table so that the entry in its a-th column and b-th row is the reduced version of the fraction a/b:

	1	2	3	4	5	6	7
1	1	2					
2	$\frac{1}{2}$						
3							
4							
5							
6							
7							

Create a list of the positive rational numbers as follows: Follow the snake path starting in the upper-left corner of the box. Each time you reach a fraction, if that number is not in your list already, add it to your list. Thus, your list will start: $1, 2, 1/2, 1/3, 3, \ldots$. Continue until you get to the seventh diagonal. Imagine that the table extends infinitely in both directions so that you may continue the list indefinitely. Does your list then contain every positive rational number exactly once? What does this say about the cardinality of $\mathbb{Q}_{>0}$?

Solution.

	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$
3	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2	$\frac{7}{3}$
4	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$
5	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1	$\frac{6}{5}$	$\frac{7}{5}$
6	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1	$\frac{7}{6}$
7	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{1}{7}$	1

$$1, 2, \frac{1}{2}, \frac{1}{3}, 3, 4, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, 5, 6, \frac{5}{2}, \frac{4}{3}, \frac{3}{4}, \frac{2}{5}, \frac{1}{6}, \frac{1}{7}, \frac{3}{5}, \frac{5}{3}, 7.$$

This extended list contains every element of $\mathbb{Q}_{>0}$ exactly once. Hence $\mathbb{Q}$ is countably infinite, and $|\mathbb{N}| = |\mathbb{Q}_{>0}|$. $\square$

PROBLEM 6.

(a) Suppose sets X and Y are countably infinite. List the elements of X and Y:

$$X: \; x_0, x_1, x_2, \ldots$$
$$Y: \; y_0, y_1, y_2, \ldots$$

Show that $X \cup Y$ is countably infinite. (Thus, X and Y are "listable". Create a list for $X \cup Y$. At first, you might assume that X and Y are disjoint, and then consider how to modify your list if $X \cap Y \neq \emptyset$.)

(b) Above, we have shown that $\mathbb{Q}_{>0}$ is countably infinite. Argue that $\mathbb{Q}$ is countably infinite.

Solution.

(a) Create the list

$$x_0, y_0, x_1, y_1, x_2, y_2, \ldots$$

Next read the list from the start, removing any elements that are repeated (these will be exactly the elements of $X \cap Y$).

(b) Using what we just showed, we can list the elements of $\mathbb{Q}_{<0} \cup \mathbb{Q}_{0>0}$, and then prepend a 0. This gives a list of $\mathbb{Q} = \{0\} \cup \mathbb{Q}_{<0} \cup \mathbb{Q}_{>0}$.