PROBLEM 1.

(a) If F is a field and x is a nonzero element of F. What is the meaning of $\frac{1}{x}$ (also denoted x^{-1})?

(b) What is $\frac{1}{3}$ in the field $\mathbb{Z}/7\mathbb{Z}$? (Denote the equivalence classes for $\mathbb{Z}/7\mathbb{Z}$ be $\{0, 1, 2, 3, 4, 5, 6\}$, for convenience.)

(c) Show that 2 does not have a multiplicative inverse in $\mathbb{Z}/8\mathbb{Z}$.

Solution.

(a) By $\frac{1}{x}$, we mean the multiplicative inverse of x, i.e., the element of F which when multiplied by x gives the multiplicative identity, 1.

(b) We have $\frac{1}{3} = 5$ in $\mathbb{Z}/7\mathbb{Z}$ since $3 \cdot 5 = 15 = 1 \bmod 7$.

(c) Here are the multiples of 2 modulo 8:

n	0	1	2	3	4	5	6	7
$2n$	0	2	4	6	0	2	4	6

So there is no element of $n \in \mathbb{Z}/8\mathbb{Z}$ such that $2n = 1$ in $\mathbb{Z}/8\mathbb{Z}$.

PROBLEM 2. Let F be a field. In the lecture notes, we proved that for any $x \in F$, we have $x \cdot 0 = 0$. Using that proof as a model, prove that if $x, y, z \in F$, then

$$z + x = z + y \implies x = y.$$

Your proof should proceed by using one axiom per step. (You will need A4, A2, and the definition of 0 (A3).) The above result is called the *cancellation law for addition* in a field.

Proof. Since F is a field, the element z has an additive inverse $-z$. Thus,

$$\begin{aligned} z + x = z + y \quad &\Rightarrow \quad -z + (z + x) = -z + (z + y) \\ &\Rightarrow \quad (-z + z) + x = (-z + z) + y \qquad \text{(associativity of +)} \\ &\Rightarrow \quad 0 + x = 0 + y \qquad \text{(definition of } -z\text{)} \\ &\Rightarrow \quad x = y \qquad \text{(definition of 0).} \end{aligned}$$

□

PROBLEM 3. Let F be a field, and let $x \in F$

(a) What is the meaning of $-x$?

(b) What is -3 in the field $\mathbb{Z}/7\mathbb{Z}$? (Again, denote the equivalence classes for $\mathbb{Z}/7\mathbb{Z}$ be $\{0, 1, 2, 3, 4, 5, 6\}$.)

(c) Prove that $-1 \cdot x = -x$. (You will need to focus on the definitions of -1 and $-x$. Since F is a field, it has a multiplicative identity 1, and that multiplicative identity must, like all element of F, have an additive inverse, -1. By definition, -1 is the element of F which when added to 1 yields the additive identity, 0. To test if a field element is $-x$, you add it to x and see if you get 0. You will also probably use the fact that $0 \cdot x = 0$, which we proved in the lecture notes.)

Solution.

(a) By $-x$, we mean the additive inverse of x, that is, the element of F which when added to x yields the additive identity, 0.

(b) We have $-3 = 4$ in $\mathbb{Z}/7\mathbb{Z}$ since $3 + 4 = 0$ in $\mathbb{Z}/7\mathbb{Z}$.

(c)

Proof. We have

$$\begin{aligned}
-1 \cdot x + x &= -1 \cdot x + 1 \cdot x && \text{(definition of 1)} \\
&= (-1 + 1) \cdot x && \text{(distributivity)} \\
&= 0 \cdot x && \text{(definition of } -1\text{)} \\
&= 0 && \text{(result from the lecture notes).}
\end{aligned}$$

Since adding $-1 \cdot x$ to x yields 0, it follows definition of the additive inverse that $-1 \cdot x = -x$. □