PROBLEM 1. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$.

(a) Give an example of two subsets X and Y of A and a function $f\colon A \to B$ such that $f(X \cap Y) \neq f(X) \cap f(Y)$.

(b) Is it possible to find subsets X and Y of A such that $f(X \cap Y) \not\subseteq f(X) \cap f(Y)$?

SOLUTION:

(a) For one example, let $f(1) = f(2) = a$ and $f(3) = b$, and let $X = \{1\}$ and $Y = \{2\}$. Then $f(X \cap Y) = f(\emptyset) = \emptyset$, while

$$f(X) \cap f(Y) = \{a\} \cap \{a\} = \{a\}\,.$$

(b) This is not possible. We proved that $f(X \cap Y) \subseteq f(X) \cap f(Y)$ for all functions f and subsets X, Y of the domain of f in the lecture.

PROBLEM 2. For each of the following functions, state why there is no inverse, or describe the inverse function.

(a)

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto |x|. \end{aligned}$$

(b)

$$\begin{aligned} g\colon \mathbb{R}_{\geq 0} &\to \mathbb{R} \\ x &\mapsto |x|. \end{aligned}$$

(c)

$$\begin{aligned} h\colon \mathbb{R}_{\geq 0} &\to \mathbb{R}_{\geq 0} \\ x &\mapsto |x|. \end{aligned}$$

SOLUTION:

(a) This function has no inverse since it is not bijective. It's not surjective since, for example, $-1 \notin \operatorname{im}(f)$. It's not injective since, for example, $|-1| = |1|$.

(b) This function has no inverse since it is not bijective. It's not surjective since, for example, $-1 \notin \operatorname{im}(g)$.

(c) The inverse of this function is

$$\begin{aligned} h^{-1}\colon \mathbb{R}_{\geq 0} &\to \mathbb{R}_{\geq 0} \\ x &\mapsto x. \end{aligned}$$

PROBLEM 3. Show that the function

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto 3x + 1 \end{aligned}$$

is a bijection by providing its inverse function. (Demonstrate that the function you produce is actually the inverse of f. You need to check both possible compositions are the identity.) [This is the second method of proving that a function is bijective. The first, which more closely follows the definition of bijectivity is to prove that the function is both injective and surjective.]

SOLUTION: The inverse function is

$$\begin{aligned} g\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto (x-1)/3. \end{aligned}$$

To verify that g is the inverse, we check

$$(f \circ g)(x) = f(g(x)) = f((x-1)/3) = 3\,((x-1)/3) + 1 = (x-1) + 1 = x,$$

and

$$(g \circ f)(x) = g(f(x)) = g(3x+1) = ((3x+1)-1)\,/3 = 3x/3 = x.$$

PROBLEM 4. Consider the functions $f(x) = x + 1$ and $g(x) = 3x$, both with domain and codomain $\mathbb{R}$. Compute the following: (i) $g \circ f$, (ii) $(g \circ f)^{-1}$, (iii) f^{-1}, (iv) g^{-1}, and (v) $f^{-1} \circ g^{-1}$. Verify that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

SOLUTION:

(i) We have

$$(g \circ f)(x) = g(f(x)) = g(x+1) = 3(x+1) = 3x + 3.$$

(ii) To find $(g \circ f)^{-1}$, we set $y = 3x+3$ and solve for x. We find $x = (y-3)/3$. Therefore, $(g \circ f)^{-1}(x) = (x-3)/3$ (with domain and codomain equal to $\mathbb{R}$).

(iii), (iv) Similarly, we find

$$f^{-1}(x) = x - 1 \quad \text{and} \quad g^{-1}(x) = x/3.$$

(v) Therefore,

$$(f^{-1} \circ g^{-1})(x) = f^{-1}(x/3) = x/3 - 1 = (x-3)/3 = (g \circ f)^{-1}.$$