PROBLEM 1. Let $A := \{1, 2, 3, 4\}$ and $B := \{a, b, c\}$. Define $f: A \to B$ by $f(1) = f(3) = a$, $f(2) = b$, and $f(4) = c$.

(a) What are the domain and codomain of f?

(b) What is the formal definition of f as a relation (a subset of $A \times B$)?

(c) Is f injective? surjective? bijective?

SOLUTION:

(a) The domain is A and the codomain is B.

(b) Formally, f is defined by its graph:

$$\{(1, a), (2, b), (3, a), (4, c)\}.$$

(c) Since $f(1) = f(3)$, the function is not injective. However, f is surjective since its image is B: 1 is in the pre-image of a (as is 3), 2 is the pre-image of b and 4 is the pre-image of c. Since f is not injective, it is not a bijection.

PROBLEM 2. Consider the absolute value function:

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto |x|. \end{aligned}$$

(a) Draw the graph of f.

(b) What is $\mathrm{im}(f)$, the image of f?

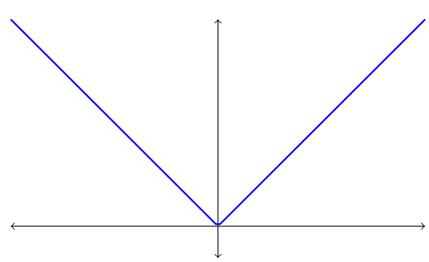
(c) Is f injective? (Prove or provide a concrete counterexample.)

(d) Is f surjective? (Prove or provide a concrete counterexample.)

(e) How are the answers to the last two questions reflected in your drawing of the graph of f?

SOLUTION:

(a)



Graph of f.

(b) The image of f is $\mathbb{R}_{\geq 0}$.

(c) The function f is not injective since, for instance $f(1) = f(-1)$.

(d) The function f is not surjective since $\mathrm{im}(f) = \mathbb{R}_{\geq 0} \subsetneq \mathrm{codomain}(f) = \mathbb{R}$. For instance, -1 is not in the image of f.

(e) We can see from the graph that f is not injective since there are some horizontal lines that meet the graph in two points.

(f) We can see from the graph that f is not surjective since there are some horizontal lines that meet the graph in no points.

PROBLEM 3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x - 7$. Prove that f is bijective. (Follow the template.)

SOLUTION:

Proof. To see that f is injective, let $x, y \in \mathbb{R}$ and suppose that $f(x) = f(y)$. It follows that $3x - 7 = 3y - 7$. Adding 7 to both sides of this equation gives $3x = 3y$. Then dividing by 3 gives $x = y$. Hence, f is injective.

To see that f is surjective, let $z \in \mathbb{R}$ (i.e. fix an arbitrary point in the codomain of f). We need to find $x \in \mathbb{R}$ (in the domain of f) such that $f(x) = z$. In other words, we need to solve the equation

$$3x - 7 = z$$

for x. We find $x = (z + 7)/3$. Check:

$$f((z + 7)/3) = 3(z + 7)/3 - 7 = z + 7 - 7 = z,$$

as required. Hence, f is surjective.

Since f is injective and surjective, it is a bijection. □