PROBLEM 1. Fill in the following addition and multiplication tables (using standard representatives for equivalence classes for convenience, e.g, 3 instead of [3]).

$\mathbb{Z}/5\mathbb{Z}$

+	0	1	2	3	4
0					
1					
2					
3					
4					

·	0	1	2	3	4
0					
1					
2					
3					
4					

$\mathbb{Z}/6\mathbb{Z}$

+	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

·	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

SOLUTION:

$\mathbb{Z}/5\mathbb{Z}$

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$\mathbb{Z}/6\mathbb{Z}$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	5	2	0	4	2
5	0	5	4	3	2	1

PROBLEM 2. Why are all of the tables in the previous problem symmetric about the diagonal from top-left to bottom-right? Do you see any other patterns?

SOLUTION: That is because addition and multiplication in $\mathbb{Z}/n\mathbb{Z}$ are commutative: $[a]+[b] = [b]+[a]$ and $[a][b] = [b][a]$. Said another way, $a+b = b+a \bmod n$ and $ab = ba \bmod n$.

Some other patterns:

(a) The rows of the addition table are cyclic shifts of each other.

(b) The equivalence classes for 0 and 1 behave as one might expect with respect to addition and multiplication.

(c) In the bottom row and last column of the multiplication tables, the classes besides [0] are listed in reverse order.

PROBLEM 3. Let $a, b \in \mathbb{Z}$. When is $a = b \bmod 2$? When is $a = b \bmod 1$? When is $a = b \bmod 0$? List the equivalence classes in each case, i.e., the elements of $\mathbb{Z}/n\mathbb{Z}$, for $n = 2, 1, 0$.

SOLUTION: We have $a = b \bmod 2$ exactly when a and b are both even or if they are both odd. There are two equivalence classes, [0] and [1].

We have $a = b \bmod 1$ when $a - b = 1 \cdot k$ for some $k \in Z$. But there is always such a k, namely, $k = a - b$. Thus, $a = b \bmod 1$ holds for all a and b. There is one equivalence class, [0].

We have $a = b \bmod 0$ when $a - b = 0 \cdot k$, i.e., when $a - b = 0$. So a and b are equal modulo 0 if and only if $a = b$. The equivalence classes are $[a]$ for $a \in \mathbb{Z}$, and each equivalence class just contains one element.

Problem 4. Use modular arithmetic to find the last two digits of the following two numbers:

$$101^{(10^{1000}+2021)} \quad \text{and} \quad 99^{(10^{1000}+2021)}.$$

Solution: To find the last two digits, we find standard representatives modulo 100. Now $101 = 1 \bmod 100$. So

$$101^{(10^{1000}+2021)} = 1^{(10^{1000}+2021)} = 1 \bmod 100.$$

The last two digits are 01. Similarly, $99 = -1 \bmod 100$. So

$$99^{(10^{1000}+2021)} = (-1)^{(10^{1000}+2021)} = -1 = 99 \bmod 100,$$

since $10^{10^{1000}+2021}$ is odd. The last two digits in this case are 99.

Problem 5 (Challenge). Let $a_1 = 3$, and for $n > 0$, define $a_n = 3^{a_{n-1}}$. Thus, $a_2 = 3^3 = 27$, and $a_3 = 3^{3^3} = 3^{27}$. What is the last digit of a_{100}? (Hint: start by considering the last digits of 3, 3^2, 3^3, 3^4, etc., until you see a pattern. You may start to think that the number 4 is significant.)

Solution: To find the last digit, work modulo 10. We have $3^4 = 81 = 1 \mod 10$. Therefore, the last digit of 3^n for any n is determined by the equivalence class of n modulo 4. For instance,

$$3^{n+4} = 3^n \cdot 3^4 = 3^n \bmod 10.$$

So to find a_{100}, we need to find a_{99} modulo 4. Since $3 = -1 \bmod 4$, we have

$$a_{99} = 3^{a_{98}} = (-1)^{a_{98}} = -1 = 3 \bmod 4$$

since a_{98} is odd. Then

$$a_{100} = 3^{a_{99}} = 3^3 = 27 = 7 \bmod 10.$$