Problem 1. Let A be the set of all lines in the plane. Is the relation "is parallel to" on A an equivalence relation? If not, which properties prevent if from being so. (Take "parallel" to mean "same slope" rather than "non-intersecting". How does this affect your answer?)

Solution: This is an equivalence relation but would not be if we took "parallel" to mean "non-intersecting".

Reflexivity. A line has the same slope as itself. Hence, the relation is reflexive. If, on the other hand, we took "parallel" to mean "non-intersecting", then reflexivity would not hold.

Symmetry. If line L is parallel to line M, then line M is parallel to line L. Hence, the relation is symmetric.

Transitivity. If L, M, N are lines and L is parallel to M and M is parallel to N, then L is parallel to N. Hence, the relation is transitive. If we took "parallel" to mean non-intersecting, we would not have transitivity: consider the case where $L = N$.

Problem 2. Let A be the set of all lines in the plane. Is the relation "is perpendicular to" on A an equivalence relation? If not, which properties prevent if from being so.

Solution: This is not an equivalence relation.

Reflexivty. A line is not perpendicular to itself.

Symmetry. If line L is perpendicular to line M, then M is perpendicular to L. Hence, the relation is symmetric.

Transitivity. Let L, M, N be lines. If L is perpendicular to M and M is perpendicular to N, then L and N are parallel, not perpendicular. Hence, the relation is not transitive.

Problem 3. For $a, b \in \mathbb{Z}$, say $a \sim b$ if $a - b = 2k$ for some $k \in \mathbb{Z}$. In other words, $a \sim b$ if $a - b$ is an even integer. Prove that $\sim$ is an equivalence relation on $\mathbb{Z}$ following the template below:

Theorem. Define a relation $\sim$ on a set A by blah, blah, blah. Then $\sim$ is an equivalence relation.

Proof. Let $a, b, c \in A$.

Reflexivity. We have $a \sim a$ since blah, blah, blah. Therefore, $\sim$ is reflexive.

Symmetry. Suppose that $a \sim b$. Then, blah, blah, blah. It follows that $b \sim a$. Therefore $\sim$ is symmetric.

Transitivity. Suppose that $a \sim b$ and $b \sim c$. Then blah, blah, blah. It follows that $a \sim c$. Therefore, $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric, and transitive, it follows that $\sim$ is an equivalence relation. $\square$

Solution:

Proof. Let $a, b, c \in \mathbb{Z}$.

Reflexivity. For each $a \in \mathbb{Z}$, we have $a \sim a$ since $a - a = 0 = 2 \cdot 0$, i.e., $a - a$ is even. Therefore, $\sim$ is reflexive.

Symmetry. Suppose that $a \sim b$. Then, $a - b = 2k$ for some $k \in \mathbb{Z}$. But then, $b - a = 2(-k)$. It follows that $b \sim a$. Therefore $\sim$ is symmetric.

Transitivity. Suppose that $a \sim b$ and $b \sim c$. Then $a - b = 2k$ and $b - c = 2k'$ for some $k, k' \in \mathbb{Z}$. But then

$$a - c = (a - b) + (b - c) = 2k - 2k' = 2(k - k').$$

It follows that $a \sim c$. Therefore, $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric, and transitive, it follows that $\sim$ is an equivalence relation. □