PROBLEM 1. Compute $\sum_{k=-2}^{2}(3k+2)$ and show that it equals $3\sum_{k=-2}^{2}k+\sum_{k=-2}^{2}2$.

SOLUTION: We have

$$\begin{aligned}\sum_{k=-2}^{2}(3k+2) &= (3(-2)+2)+(3(-1)+2)+(3(0)+2)+(3(1)+2)+(3(2)+2)\\ &= -4-1+2+5+8\\ &= 10.\end{aligned}$$

On the other hand,

$$\begin{aligned}3\sum_{k=-2}^{2}k+\sum_{k=-2}^{2}2 &= 3(-2-1+0+1+2)+(2+2+2+2+2)\\ &= 3(0)+10\\ &= 10.\end{aligned}$$

PROBLEM 2. Use induction to prove that each $n \geq 1$,

$$1\cdot 2+2\cdot 3+\cdots+n\cdot(n+1)=\frac{n(n+1)(n+2)}{3}.$$

SOLUTION: We will prove this by induction. The base case, $n=1$, holds since

$$1\cdot 2=\frac{1(1+1)(1+2)}{3}.$$

Now suppose that the result holds for some $n \geq 1$. It follows that

$$\begin{aligned}1\cdot 2+2\cdot 3+\cdots+(n+1)(n+2) &= (1\cdot 2+2\cdot 3+\cdots+n(n+1))+(n+1)(n+2)\\ &= \frac{n(n+1)(n+2)}{3}+(n+1)(n+2) \qquad \text{(by the induction hypothesis)}\\ &= (n+1)(n+2)\left(\frac{n}{3}+1\right) \qquad \text{(factoring)}\\ &= \frac{(n+1)(n+2)(n+3)}{3}\\ &= \frac{(n+1)((n+1)+1)((n+1)+2)}{3}.\end{aligned}$$

The result then holds for the case $n+1$, as well. The result follows by induction.

Alternative solution. We will prove this by induction. The base case, $n=1$, holds since

$$\sum_{k=1}^{1}k(k+1)=1(1+1)=2=\frac{1(1+1)(1+2)}{3}.$$

Now suppose that the result holds for some $n \geq 1$. It follows that

$$\begin{aligned}
\sum_{k=1}^{n+1} k(k+1) &= \sum_{k=1}^{n} k(k+1) + (n+1)(n+2) \\
&= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \qquad \text{(by the induction hypothesis)} \\
&= (n+1)(n+2)\left(\frac{n}{3}+1\right) \qquad \text{(factoring)} \\
&= \frac{(n+1)(n+2)(n+3}{3} \\
&= \frac{(n+1)((n+1)+1)((n+1)+2)}{3}.
\end{aligned}$$

The result then holds for the case $n+1$, as well. The result follows by induction. ☐

PROBLEM 3. Let $a > -1$ be a real number. Use induction to show that for all integers $n \geq 0$,

$$(1+a)^n \geq 1 + na.$$

(Note: for any nonzero real number x, we have that $x^0 = 1$, by definition.)

SOLUTION: We will prove this by induction. The base case, $n = 0$, holds since

$$(1+a)^0 = 1 \geq 1 = 1 + 0 \cdot a.$$

Suppose the result holds from some $n \geq 0$. Then

$$\begin{aligned}
(1+a)^{n+1} &= (1+a)^n(1+a) \\
&\geq (1+na)(1+a) \qquad \text{(by the induction hypothesis} \\
&\qquad\qquad\qquad \text{and the fact that } 1+a > 0) \\
&= 1 + (n+1)a + na^2 \\
&\geq 1 + (n+1)a \qquad \text{(since } na^2 \geq 0).
\end{aligned}$$

Thus, the result then holds for $n+1$, too. The result holds for all $n \geq 0$ by induction.