Math 111

9/7/09

No quiz this week.

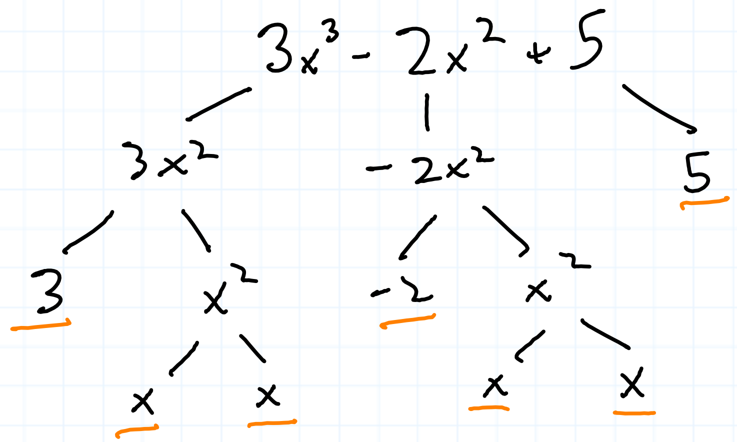
Last time: $\lim_{x \to 5} x^2 = 25$

What about $\lim_{x \to 2} x^3 = 8$ or $\lim_{x \to 1} x^3 + 2x + 5 = 8$ or $\lim_{x \to 1} \frac{x^4 + 2x + 1}{x^3 + 7} = \frac{1}{2}$?

The straightforward approach gets increasingly difficult.

Idea: Complicated functions are sometimes built from simpler functions.

Example:



Strategy

1. Find limits of simple functions.
2. Show that limits behave nicely with respect to "building operations."

Two easy limits

<u>Proposition</u> Let c be any real number.

1. Let a be a constant. Then $\lim_{x \to c} a = a$.

2. $\lim_{x \to c} x = c$.

Pf/ 1. Let $f(x) = a$. Given $\varepsilon > 0$, we have

$$|f(x) - a| = |a - a| = 0 < \varepsilon$$

for <u>any</u> x. So pick an arbitrary δ, say $\delta = 1$. If $0 < |x - c| < \delta$, then $|f(x) - a| < \varepsilon$, as required.

2. Now let $f(x) = x$. Given $\varepsilon > 0$, let $\delta = \varepsilon$. Suppose $0 < |x - c| < \delta = \varepsilon$. Then

$$|f(x) - c| = |x - c| < \delta = \varepsilon. \qquad \square$$

Limit Theorems (pp. 59–60, Appendix A2–A3)

Suppose $\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} g(x) = M$. Then

(a) $\lim\limits_{x \to c} f(x) + g(x) = L + M$

(b) $\lim\limits_{x \to c} f(x)g(x) = LM$

(c) $\lim\limits_{x \to c} \dfrac{1}{g(x)} = \dfrac{1}{M}$ if $M \neq 0$

(d) $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ if $M \neq 0$.

Example of use:

Let $f(x) = x$ and $g(x) = x$. By our earlier proposition (part 2), $\lim\limits_{x \to 5} f(x) = 5$ and $\lim\limits_{x \to 5} g(x) = 5$. Now $f(x)g(x) = x^2$. So (b) says

$$\lim_{x \to 5} x^2 = \lim_{x \to 5} f(x)g(x) = 5 \cdot 5 = 25.$$

More succinctly:

$$\lim_{x\to 5} x^2 = \left(\lim_{x\to 5} x\right)\left(\lim_{x\to 5} x\right) \qquad \text{(by (b))}$$
$$= 5 \cdot 5 \qquad \text{(by the Proposition, part 2)}$$
$$= 25.$$

Example

$$\lim_{x\to 1} 5x^3 - 3x + 2 = 4$$

Proof/ By our limit theorems,

Limit Theorem

$$\lim_{x\to 1} 5x^3 - 3x + 2 = \lim_{x\to 1} 5x^3 + \lim_{x\to 1} (-3x) + \lim_{x\to 1} 2 \qquad \text{(LT (a))}$$
$$= \left(\lim_{x\to 1} 5\right)\left(\lim_{x\to 1} x^3\right) + \left(\lim_{x\to 1} -3\right)\left(\lim_{x\to 1} x\right) + \lim_{x\to 1} 2 \qquad \text{(LT (b))}$$
$$= 5 \lim_{x\to 1} x^3 + (-3)(1) + 2 \qquad \text{(standard limits)}$$

$$= 5\left(\lim_{x\to1} x\right)\left(\lim_{x\to1} x^2\right) - 1$$

$$= 5\left(\lim_{x\to1} x\right)\left(\lim_{x\to1} x\right)\left(\lim_{x\to1} x\right) - 1$$

$$= 5\cdot1\cdot1\cdot1 - 1 = 5-1 = 4. \quad \square$$

Example

$$\lim_{x\to2}\frac{x^3+5x+3}{x+4} = \frac{\lim_{x\to2} x^3+5x+3}{\lim_{x\to2} x+4} \qquad (\text{LT (d)})$$

$$= \frac{\left(\lim_{x\to2} x\right)\left(\lim_{x\to2} x\right)\left(\lim_{x\to2} x\right) + \lim_{x\to2} 5 \lim_{x\to2} x + \lim_{x\to2} 3}{\lim_{x\to2} x + \lim_{x\to2} 4} \qquad (\text{LT (a)} + \text{(b)})$$

$$= \frac{2\cdot2\cdot2 + 5\cdot2 + 3}{2+4}$$

$$= \frac{8+10+3}{6} = \frac{21}{6} = \frac{7}{2}.$$

Next time: Prove the limit theorems