Math 111 11/11/09

Quiz State and prove the fundamental theorem of calculus (version 1).

More on anti-derivatives

We've seen that $[\ln(x)]' = \frac{1}{x}$, but since $\ln(x)$ is only defined for $x > 0$, this formula only holds for $x > 0$. What about the case $x < 0$?

Prop. For $x \neq 0$, $(\ln|x|)' = \frac{1}{x}$.

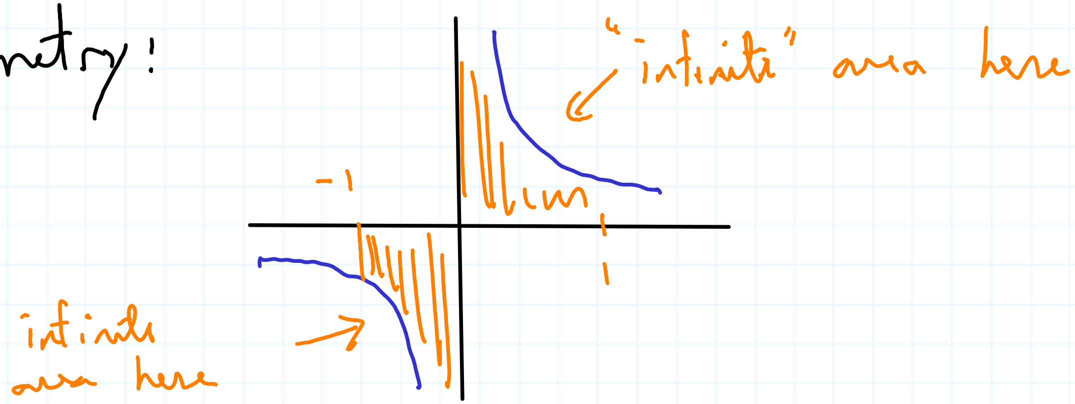
Pf/ If $x > 0$, then $|x| = x$, and we've already seen $(\ln x)' = \frac{1}{x}$ in this case. If $x < 0$, then $(\ln|x|)' = [\ln(-x)]' = \frac{1}{-x}(-x)'$
$= \frac{1}{-x}(-1) = \frac{1}{x}$ (by the chain rule). $\square$

Cor. $\int_a^b \frac{1}{x}\,dx = \ln|b| - \ln|a|$ for $0 < a < b$ or $a < b < 0$.

Beware: One might think

$$\int_{-1}^{1} \frac{1}{x}\,dx = \ln|x|\,\Big|_{x=-1}^{1} = \ln|1| - \ln|-1| = \ln 1 - \ln 1 = 0.$$

But in fact $\int_{-1}^{1} \frac{1}{x}\,dx$ is not defined since $\frac{1}{x}$ is unbounded on $[-1,1]$. Consider the geometry!



Some derivatives and integrals involving logs

* $$\left(\ln\sqrt{x^3+2x}\right)' = \left(\ln (x^3+2x)^{\frac{1}{2}}\right)'$$

$$= \frac{1}{(x^3+2x)^{\frac{1}{2}}} \cdot \frac{1}{2}(x^3+2x)^{-\frac{1}{2}}(3x^2+2)$$

$$= \frac{3x^2+2}{2(x^3+2x)}.$$

* Easier: $$\left(\ln\sqrt{x^3+2x}\right)' = \left(\ln(x^3+2x)^{\frac{1}{2}}\right)'$$

$$= \left[\frac{1}{2}\ln(x^3+2x)\right)'$$

$$= \frac{1}{2}\,\frac{1}{x^3+2x}\cdot(3x+2).$$

Since $x^2+4>0$

* $$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|x^2+4| + c = \frac{1}{2}\ln(x^2+4) + c = \ln(x^2+4)^{\frac{1}{2}} + c$$

$$= \ln\sqrt{x^2+4} + c.$$

$u = x^2+4$, $du = 2x\,dx$

* $$\int \frac{x^2-1}{x^3-3x+1}\,dx = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| = \frac{1}{3}\ln|u|+c$$

$$= \frac{1}{3}\ln|x^3-3x+1|+c$$

$u = x^3-3x+1$, $du = (3x^2-3)dx = 3(x^2-1)dx$

Antiderivatives of trig. functions

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,du = -\int \frac{du}{u} = -\ln|u|+c = -\ln|\cos x|+c$$

$u = \cos x$, $du = -\sin x\,dx$

$$= \ln|(\cos x)^{-1}|+c$$

$$= \ln|\sec(x)|+c.$$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{du}{u} = \ln|u|+c = \ln|\sin x|+c.$$

$u = \sin x$, $du = \cos x\,dx$

$$\int \sec x \, dx = \int \sec x \cdot \underbrace{\frac{\sec x + \tan x}{\sec x + \tan x}}_{1} dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} = \int \frac{du}{u} = \ln|u| + c = \ln|\sec x + \tan x| + c.$$

$u = \sec x + \tan x, \quad du = (\sec x \tan x + \sec^2 x)\, dx$

$$\int \csc x \, dx = \int \csc x \, \frac{\csc x + \cot x}{\csc x + \cot x} dx$$

$$= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} = -\int \frac{du}{u} = -\ln|u| + c = -\ln|\csc x + \cot x| + c$$

$u = \csc x + \cot x, \quad du = -\csc x \cot x - \csc^2 x$

$$\left(= -\ln\left|\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right| + c \right.$$

$$\left. = -\ln\left|\frac{1+\cos x}{\sin x}\right| + c = \ln\left|\frac{\sin x}{1+\cos x}\right| + c \right).$$