Quiz.

1. Carefully state the MVT.
2. Suppose f is a differentiable function on an open interval I, and suppose $f'(x) > 0$ for all $x \in I$. Use the MVT to show that f is strictly increasing on I.

Infs and Sups

See handout

Def. Let $X \subseteq \mathbb{R}$. A number B is an upper bound for X if $b \geq x$ for all $x \in X$. A number s is the supremum for X if it is the least upper bound for X. That is, s is the supremum of X if ① $s \geq x$ for all $x \in X$ (i.e., s is an upper bound for X) and ② if B is an upper bound for X, then $s \leq B$ (i.e., s is no greater than any other upper bound).

A number L is a lower bound for X if $L \le x$ for all $x \in X$. A number t is the infimum for X if it is the greatest lower bound of X. That is, t is the infimum of X if ① $t \le x$ for all $x \in X$ (i.e., t is a lower bound for X), and ② if L is a lower bound for X, then $t \ge L$ (i.e., t is no smaller than any other lower bound). The supremum and infumum of X, if they exist, are denoted $\sup(X)$ and $\inf(X)$, respectively.

Examples

1. Let $X = \{1, 4\}$. Then $10, 10^{100}$, and 4 are upper bounds for X and $\sup(X) = 4$. The numbers $-100, -50, 0, \frac{1}{2}$, and 1 are lower bounds, and $\inf(X) = 1$.

2. $X = [0,2)$. In this case, $\inf(X) = 0$ and $\sup(X) = 2$.

NOTE carefully: The sup and inf of a set are not necessarily elements of the set. In the example above, for instance, $2 \notin [0,2)$.

3. $X = \{1, 2, 3, 4, 5, \dots\}$. In this case, $\inf(X) = 1$ and $\sup(X)$ does not exist.

4. $X = \{\frac{1}{n} : n = 1, 2, 3, \dots\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$. In this case, $\inf(X) = 0$ and $\sup(X) = 1$.

5. $X = \{\frac{n}{n+1} : n = 1, 2, 3, \dots\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\}$. In this case, $\inf(X) = \frac{1}{2}$ and $\sup(X) = 1$.

Thm. Let $X \subseteq \mathbb{R}$ be a non-empty subset. If X is bounded above, then $\sup(X)$ exists. If X is bounded below, then $\inf(X)$ exists.

Prop. Suppose $\emptyset \neq X \subseteq Y \subseteq \mathbb{R}$.

(1) If $\sup(Y)$ exists, then so does $\sup(X)$ and $\sup(X) \leq \sup(Y)$

(2) If $\inf(X)$ exists, then so does $\inf(Y)$ and $\inf(X) \leq \inf(Y)$.

Pf/ (1) We first show that X is bounded above by $\sup(Y)$. Let $x \in X$. Then since $X \subseteq Y$, we have $x \in Y$, and hence $x \leq \sup(Y)$. Next, since X is a non-empty subset of $\mathbb{R}$, bounded above, $\sup(X)$ exists. Finally, since $\sup(Y)$ is an upper bound for X and $\sup(X)$ is the least upper bound for X, we have $\sup(X) \leq \sup(Y)$.

(2) Similar. $\square$