Exploring entanglement with the help of quantum state measurement

E. Dederick and M. Beck[a)]
Department of Physics, Whitman College, Walla Walla, Washington 99362

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We have performed a series of experiments using a spontaneous parametric down-conversion source to produce pairs of photons in either entangled or non-entangled polarization states. We determine the full quantum mechanical polarization state of one photon, conditioned on the results of measurements performed on the other photon. For non-entangled states, we find that the measured state of one photon is independent of measurements performed on the other. However, for entangled states, the measured state does depend on the results of measurements performed on the other photon. This is possible because of the nonlocal nature of entangled states. These experiments are suitable for an undergraduate teaching laboratory. *© 2014 American Association of Physics Teachers.*
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I. INTRODUCTION

In quantum mechanics, the information about a quantum system is contained in its state. For a system prepared in a pure state (a state that is the same on every trial) the state may be described by a state vector, or ket, $|\psi\rangle$. However, in real experiments, it is usually the case that the state preparation procedure is not perfect, so the system is not always prepared in the same pure state $|\psi\rangle$ on each trial. In such cases, the system is said to be in a mixed state, which can be described in terms of a density operator $\hat{\rho}$.[1] For systems with a discrete basis, the density operator may be represented as a matrix, often referred to as the density matrix.

The state of a quantum system encompasses *everything* that is knowable about that system. In this sense, the state represents ultimate knowledge about the system. If one is able to measure or otherwise determine the state of a system, the known state can be used to calculate any quantity of interest. For this reason, quantum state measurement has become an important tool for physicists studying quantum information.[2–5] Quantum state measurement is often referred to as a quantum state tomography, because the original algorithms used to determine the state were the same as those used in tomographic imaging.[6]

Classical particles may be correlated in a manner that allows measurements in one location to determine results of measurements in another location. For classical particles, however, perfect correlations can exist only in a single measurement basis. If measurements are performed in a different basis the classical correlations are reduced. For example, if classical polarization measurements are perfectly correlated in the horizontal-vertical basis, they will not be correlated in the circular-polarization basis. Quantum particles can be placed in entangled states that have correlations that are stronger than those allowed by classical physics (for entangled particles perfect correlations can remain for measurements performed in any basis). Entanglement is what leads to uniquely quantum mechanical phenomena such as violations of local realism, quantum teleportation, and quantum computing.[1,7]

Undergraduate experiments involving entangled particles have been previously reported.[1,8–12] Results of these experiments imply that measurements performed on one particle can change the state of another particle, even if the particles are physically separated. It is this property of entangled particles that so beguiles physicists, and that led Einstein to refer to "spooky actions at a distance."[13] However, in these previous experiments, the results are often subtle and not fully appreciated by non-experts. In order to unequivocally demonstrate that measurements performed on one particle can actually change the state of the other, it is best to explicitly measure the state. Here, we are able to accomplish this by measuring the quantum state of one photon, conditioned upon the results of measurements performed on another photon.

II. THEORY

Before discussing the experiments, we will present the theory behind them. We begin by describing the polarization states we use in the experiments. Then we provide some background on the density operator, because this is what we determine in the experiments. In particular, we show how the results of measurements affect the density operator and how one obtains the density operator of a single particle from the density operator of a two-particle system. Finally, we present the theory of quantum state measurement.

A. Polarization states

The polarization state of a single photon can be written in terms of basis states corresponding to horizontal $|H\rangle$ and vertical $|V\rangle$ polarizations. A general elliptical polarization state is given by

$$|e\rangle = a|H\rangle + be^{i\phi}|V\rangle, \tag{1}$$

where a, b, and ϕ are real numbers, and normalization requires $a^2 + b^2 = 1$. Important special cases are the $\pm 45°$ linear polarization states

$$|+45\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle) \tag{2}$$

and

$$|-45\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle), \tag{3}$$

and the left- and right-circular polarization states

$$|L\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle) \tag{4}$$

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and

$$|R\rangle = \frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle). \tag{5}$$

In our experiments, we use a spontaneous parametric down-conversion (SPDC) source that produces two photons referred to as the signal and idler. Our SPDC source is capable of producing pairs of photons that are both horizontally polarized $|H,H\rangle$, both vertically polarized $|V,V\rangle$, or any linear combination of these two possibilities. For the experiments involving entangled photons, we adjust our source to produce photons in the Bell state

$$\begin{aligned}|\phi^+\rangle &= \frac{1}{\sqrt{2}}(|H,H\rangle + |V,V\rangle)\\ &= \frac{1}{\sqrt{2}}\left(|H\rangle_s|H\rangle_i + |V\rangle_s|V\rangle_i\right).\end{aligned} \tag{6}$$

Here, we have used two different notations, the latter of which explicitly labels the polarization of each photon.

B. The density operator

Here we describe, without proof, the properties of the density operator that are needed for our discussion of quantum state measurement. More detailed information can be found in Ref. 14 or Complement 8.A of Ref. 1.

The density operator corresponding to a pure state $|\psi\rangle$ is given by

$$\hat{\rho} = |\psi\rangle\langle\psi|. \tag{7}$$

A system that is fluctuating, or that is not always prepared in the same pure state, is called a mixed state. For a mixed state in which each state $|\psi_j\rangle$ is prepared with probability p_j, the density operator is given by

$$\hat{\rho} = \sum_j p_j|\psi_j\rangle\langle\psi_j|. \tag{8}$$

The probabilities must behave like classical probabilities, meaning the p_j's are real, they all lie between zero and one ($0 \le p_j \le 1$), and they satisfy the normalization condition

$$\sum_j p_j = 1. \tag{9}$$

It is important to note that the states $|\psi_j\rangle$ are assumed to be normalized, but they need not be orthogonal nor do they need to form a basis; they are merely states that the system is prepared in with some probability.

The density matrix is the representation of the density operator in a particular basis. If the states $|\phi_n\rangle$ form an orthonormal basis, the elements of the density matrix are given by

$$\rho_{mn} = \langle\phi_m|\hat{\rho}|\phi_n\rangle. \tag{10}$$

The trace of a matrix is the sum of its diagonal elements, and we can define the trace of an operator similarly. The trace of the density operator is thus

$$\mathrm{Tr}(\hat{\rho}) = \sum_n \rho_{nn} = \sum_n \langle\phi_n|\hat{\rho}|\phi_n\rangle = 1, \tag{11}$$

where equality with 1 is the normalization condition for the density operator, which is ensured by Eq. (9). The expectation value of an operator $\hat{O}$ can be found by multiplying $\hat{O}$ by the density operator, and then computing the trace, as in

$$\langle\hat{O}\rangle = \mathrm{Tr}\left(\hat{O}\hat{\rho}\right) = \mathrm{Tr}\left(\hat{\rho}\hat{O}\right). \tag{12}$$

The density matrix describing the polarization of a photon in the horizontal/vertical basis is

$$\hat{\rho} = \begin{pmatrix} \langle H|\hat{\rho}|H\rangle & \langle H|\hat{\rho}|V\rangle \\ \langle V|\hat{\rho}|H\rangle & \langle V|\hat{\rho}|V\rangle \end{pmatrix}. \tag{13}$$

When expressing operators as matrices we will always use this basis. The density matrices of the horizontal and vertical polarization states are thus

$$\hat{\rho}_H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{14}$$

and

$$\hat{\rho}_V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{15}$$

Similarly, the density matrices for the states of Eqs. (2)–(5) are

$$\hat{\rho}_{+45} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{16}$$

$$\hat{\rho}_{-45} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{17}$$

$$\hat{\rho}_L = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \tag{18}$$

and

$$\hat{\rho}_R = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}. \tag{19}$$

For a pure state it can be shown that $\mathrm{Tr}\left(\hat{\rho}^2\right) = 1$, while for any non-pure state $\mathrm{Tr}\left(\hat{\rho}^2\right) < 1$. We can thus use $\mathrm{Tr}\left(\hat{\rho}^2\right)$ as a measure of the purity of a state. The amount of overlap between two states $\hat{\rho}_1$ and $\hat{\rho}_2$ can be expressed as the fidelity F, defined as[5]

$$F = \left[\mathrm{Tr}\left(\sqrt{\sqrt{\hat{\rho}_1}\,\hat{\rho}_2\sqrt{\hat{\rho}_1}}\right)\right]^2. \tag{20}$$

The fidelity takes on values $0 \le F \le 1$, and in the case when both states are pure it simplifies to

$$F = \mathrm{Tr}(\hat{\rho}_1\hat{\rho}_2) = |\langle\psi_1|\psi_2\rangle|^2. \tag{21}$$

C. Projective measurements

Let's assume that our system consists of two particles whose joint quantum state is given by $\hat{\rho}$. A measurement is performed on particle 1, and this measurement corresponds to an observable, represented by operator $\hat{A}$. The postulates of quantum mechanics tell us that this measurement must return an eigenvalue a of $\hat{A}$, and after the measurement particle 1 is left in the eigenstate $|a\rangle_1$ that corresponds to the measured eigenvalue. The question is, what is the state of particle 2, $\hat{\rho}_2$, after this measurement? To determine $\hat{\rho}_2$, we first use the projection operator $\hat{P}_a = |a\rangle_{11}\langle a|$ to project $\hat{\rho}$ onto the state determined by the measurement result. The state of particle 2 is then obtained by "averaging" the joint state over the state of particle 1 (this operation is called a partial trace). Since we want $\hat{\rho}_2$ to be normalized, we then renormalize the resulting state. More details of this calculation are given in the Appendix, but the result is that the state of particle 2 is given by

$$\hat{\rho}_2 = \frac{{}_1\langle a|\hat{\rho}|a\rangle_1}{\mathrm{Tr}\left({}_1\langle a|\hat{\rho}|a\rangle_1\right)}. \tag{22}$$

These concepts are best illustrated with an example. Consider the polarization-entangled Bell state $|\phi^+\rangle$ that we use in the experiments, as given in Eq. (6). The density operator corresponding to this state is

$$\hat{\rho} = |\phi^+\rangle\langle\phi^+| = \left[\frac{1}{\sqrt{2}}(|H,H\rangle + |V,V\rangle)\right] \times \left[\frac{1}{\sqrt{2}}(\langle H,H| + \langle V,V|)\right], \tag{23}$$

which can be expanded as

$$\hat{\rho} = \frac{1}{2}(|H,H\rangle\langle H,H| + |H,H\rangle\langle V,V| + |V,V\rangle\langle H,H| + |V,V\rangle\langle V,V|). \tag{24}$$

Suppose that a measurement is performed on the idler photon and it is found to be horizontally polarized. The state of the signal photon after this measurement can be determined by inserting Eq. (24) into Eq. (22). The numerator is given by

$${}_i\langle H|\hat{\rho}|H\rangle_i = \frac{1}{2}{}_i\langle H|(|H,H\rangle\langle H,H| + |H,H\rangle\langle V,V| + |V,V\rangle\langle H,H| + |V,V\rangle\langle V,V|)|H\rangle_i, \tag{25}$$

which simplifies to

$${}_i\langle H|\hat{\rho}|H\rangle_i = \frac{1}{2}\left(|H\rangle_{ss}\langle H| + 0 + 0 + 0\right) = \frac{1}{2}|H\rangle_{ss}\langle H|. \tag{26}$$

The trace of this operator [denominator of Eq. (22)] is 1/2, so the density matrix of the signal state is $\hat{\rho}_s = |H\rangle_{ss}\langle H|$. The signal photon is thus in state $|H\rangle_s$ after the measurement, which is what we would have intuitively guessed from Eq. (6).

More generally, assume that the measurement on the idler photon finds it to be elliptically polarized, corresponding to the state $|e\rangle_i$ of Eq. (1). In the Appendix, we show that the signal photon is then projected into an elliptical-polarization state that is the complex conjugate of the idler photon's state:

$$|e^*\rangle_s = a|H\rangle_s + be^{-i\phi}|V\rangle_s. \tag{27}$$

D. Quantum state measurement

A simple method for determining the polarization state of a beam of photons, assuming the state is pure, is described in Complement 5.A of Ref. 1. However, in general, the state will not be pure so we will need to determine the density operator. To do this, we will use the method described by Altepeter and coworkers in Ref. 5.

Consider a beam of photons prepared in state $\hat{\rho}$. We can perform a measurement of the polarization of this beam in the horizontal-vertical basis (HV-basis) by using a polarization analyzer PA (e.g., a beam-displacing polarizer) that splits the beam into its horizontal and vertical components (Fig. 1). If we assign the eigenvalue $+1$ to photons measured to be horizontally polarized, and -1 to photons measured to be vertically polarized, we can construct a Hermitian polarization operator $\hat{S}_1$ in terms of the projection operators onto the $|H\rangle$ and $|V\rangle$ states as

$$\hat{S}_1 = (+1)|H\rangle\langle H| + (-1)|V\rangle\langle V| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{28}$$

We can perform polarization measurements in other bases as well. To perform measurements in the $\pm 45°$ linear polarization basis, we insert a half-wave plate whose fast axis is oriented at 22.5° with respect to the horizontal before the PA in Fig. 1. This wave plate will rotate linear polarization by 45°, with the net result that $+45°$ polarized photons (eigenvalue $+1$) will be detected at one detector and $-45°$ polarized photons (eigenvalue -1) will be detected at the other. The polarization operator corresponding to measurements in this basis is

$$\hat{S}_2 = (+1)|{+45}\rangle\langle{+45}| + (-1)|{-45}\rangle\langle{-45}|, \tag{29}$$

which can be expressed as a matrix in the horizontal-vertical basis as

$$\hat{S}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{30}$$

To perform measurements in the circular polarization basis, we insert a quarter-wave plate whose fast axis is oriented at 45° with respect to the horizontal before the PA in Fig. 1. This wave plate converts linear polarization to circular polarization (and vice versa), with the net result that left-circular polarized photons (eigenvalue $+1$) will be detected at one detector and right-circular polarized photons (eigenvalue -1) will be detected at the other. The operator corresponding to these measurements is

$$\hat{S}_3 = (+1)|L\rangle\langle L| + (-1)|R\rangle\langle R| = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{31}$$

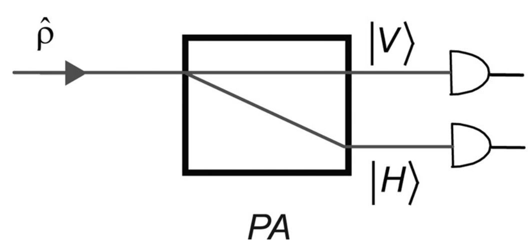

Fig. 1. A polarization analyzer (PA) that splits a beam into its horizontal and vertical components.

Finally, we will find it convenient to express the identity operator as $\hat{S}_0$:

$$\hat{S}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{32}$$

Experimentally, the expectation values $\langle \hat{S}_j \rangle$ can be calculated by simply averaging the measured values. Theoretically, the expectation values can be computed from the density operator using Eq. (12), or

$$\langle \hat{S}_j \rangle = \mathrm{Tr}\left(\hat{S}_j \hat{\rho}\right). \tag{33}$$

Note that the expectation value of the identity operator is unity:

$$\langle \hat{S}_0 \rangle = \mathrm{Tr}\left(\hat{S}_0 \hat{\rho}\right) = \mathrm{Tr}(\hat{\rho}) = 1. \tag{34}$$

Those familiar with the theory of polarization of a classical electromagnetic wave may recognize the expectation values $\langle \hat{S}_j \rangle$ as the (normalized) Stokes parameters of the beam.[15] The Stokes parameters specify the polarization of a fluctuating classical electromagnetic wave. It turns out that they also specify the quantum polarization state of a beam of photons, as it can be shown that the density operator can be written as[5]

$$\hat{\rho} = \frac{1}{2} \sum_{j=0}^{3} \langle \hat{S}_j \rangle \hat{S}_j. \tag{35}$$

Thus, the procedure for measuring the density operator describing the polarization state of a beam of photons is as follows. First, perform measurements of the polarization in the *HV*-basis using the apparatus of Fig. 1, and from these measurements determine $\langle \hat{S}_1 \rangle$. Next, insert a half-wave plate into the apparatus in order to perform measurements in the $\pm 45°$-basis and determine $\langle \hat{S}_2 \rangle$. Then replace the half-wave plate with a quarter-wave plate in order to perform measurements in the circular polarization basis and determine $\langle \hat{S}_3 \rangle$. Finally, using the matrix representations of the operators $\hat{S}_j$, Eqs. (28)–(32), and fact that $\langle \hat{S}_0 \rangle = 1$, the density matrix can be calculated using Eq. (35). It is important to note that in order for this state determination technique to work properly, the system must be prepared in the same state for all of the measurements.

III. EXPERIMENTS

A. The experimental apparatus

The experimental apparatus is shown in Fig. 2. A 100-mW, 405-nm laser diode pumps a pair of Type-I beta-barium borate (BBO) down-conversion crystals, whose axes are oriented at right angles with respect to each other. Down-converted photons make angles of 3° with respect to the pump beam and have a wavelength of approximately 810 nm. As described above, the source produces pairs of photons in an arbitrary linear combination of the $|H, H\rangle$ and $|V, V\rangle$ polarization states. The relative amplitude and phase of the states are varied by adjusting the half-wave plate and the birefringent plate that the pump laser passes through before striking the down-conversion crystals. More details regarding the experimental apparatus can be found in Refs. 1 and 16.

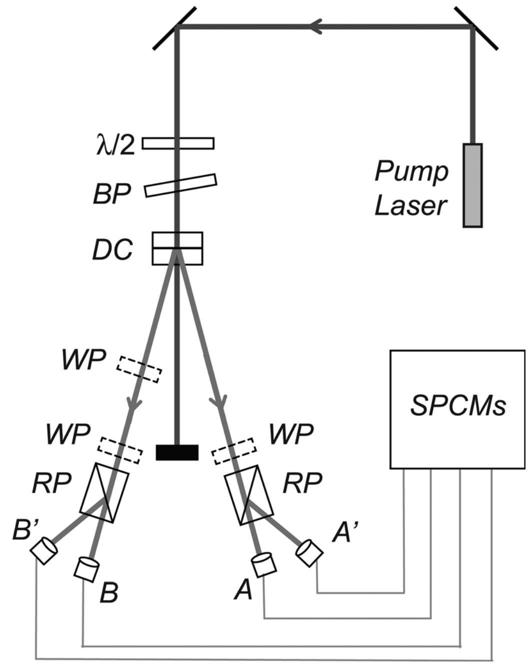

Fig. 2. A schematic of the experimental apparatus. Here $\lambda/2$ denotes a half-wave plate, *BP* denotes a birefringent plate, *DC* denotes the down-conversion crystals, *WP* denotes an optional wave plate, and *RP* denotes a Rochon polarizer. Lenses at A, A', B and B' focus the down-converted beams into multimode optical fibers, which direct the light to single photon counting modules (SPCMs).

The idler beam passes through a Rochon polarizer (*RP*) that transmits horizontally polarized photons to detector A and deflects vertically polarized photons to detector A'. A half- or quarter-wave plate may be inserted in front of the *RP* in order to project diagonally or circularly polarized light onto the two detectors. The signal beam passes through an optional half- or quarter-wave plate that can be used to modify the state of the signal beam. Then an *RP* and wave plate is used to allow the B and B' detectors to perform polarization measurements in the three different bases needed to reconstruct the density operator of the signal beam, as described in Sec. II D.

To determine the state of the signal beam conditioned on the detection of an idler photon at A, we measure the number of coincidence counts in a fixed time interval between the A and B detectors (N_{AB}) and the A and B' detectors ($N_{AB'}$). For measurements performed in the horizontal-vertical basis, we can obtain the first Stokes parameter as

$$\langle \hat{S}_1 \rangle = \frac{N_{AB} - N_{AB'}}{N_{AB} + N_{AB'}}. \tag{36}$$

By performing measurements in the two other required bases, as described above, we can determine the state of the signal beam. In a similar manner, we can determine the state of the signal beam conditioned on the detection of an idler photon at A' by measuring the coincidences $N_{A'B}$ and $N_{A'B'}$. In order to measure the state of the signal beam conditioned on the presence of an idler photon (i.e., detection at either A or A') we can simply add the A and A' coincidences and use $N_{AB} + N_{A'B}$ and $N_{AB'} + N_{A'B'}$ to determine the state.

B. Non-entangled state

First we adjust the source to produce down-converted photons in the non-entangled state $|H,H\rangle$, in which both signal and idler photons are horizontally polarized. We insert a quarter-wave plate into the signal beam in order to change the state of this beam to $|L\rangle_s$. We insert a quarter-wave plate into the idler beam so that left-circularly polarized photons are detected at A, and right-circularly polarized photons are detected at A'. (The reason for inserting this quarter-wave plate is to obtain roughly equal counts at the A and A' detectors. Without this wave plate, the horizontally polarized photons produced at the source would be detected at A, and very few photons would be detected at A'. The state reconstruction conditioned on detection at A' would be unreliable under these circumstances.)

The reconstructed states of the signal beam are shown in Fig. 3. Figure 3(a) shows the state for idler detections at A. Comparing the measured state to $\hat{\rho}_L$ [Eq. (18)], we find that the fidelity is $F = 0.96$. (Recall that the fidelity is a measure of the amount of overlap of two states, so by this measure there is 96% overlap between our measured state and $\hat{\rho}_L$.) We also find that the measured state is fairly pure, with $\mathrm{Tr}(\hat{\rho}^2) = 0.98$. As such, we can say that the measured state closely approximates the state $|L\rangle_s$, which is the state that we prepared the signal beam in and that we expected to measure.

Figure 3(b) shows the measured state of the signal beam for idler photons detected at A', and Fig. 3(c) shows the state for detections at either A or A'. The most important thing to note is that all three measured states are nearly the same. All three measured states have $\mathrm{Tr}(\hat{\rho}^2) \geq 0.94$, and compared to $\hat{\rho}_L$ they have $F \geq 0.93$.

We have also repeated this experiment, replacing the quarter-wave plate in the idler beam with a half-wave plate and projecting $+45°$ polarized photons onto detector A, and $-45°$ polarized photons onto A'. We again find that the signal beam is well described by the state $|L\rangle_s$. All of these measurements taken together indicate that for photons prepared in the state $|H,H\rangle$, the state of the signal beam is independent of measurements performed on the idler beam.

The purities and fidelities of our measured states are lower than the ideal values of 1. This is most likely due to imperfections in our state preparation procedure, which means that we are not perfectly producing the state $|H,H\rangle$. For example, any depolarization of the pump laser would mean that we would be producing some vertically polarized signal and idler photons. Also, if the two down-conversion crystals are not perfectly orthogonal, some non-horizontally polarized photons will be produced. We can improve the quality of our state production by inserting a linear polarizer oriented along the horizontal direction into the signal beam in order to better define the polarization of the signal photon. This polarizer is inserted after the down-conversion crystal (but before the quarter wave plate), which converts the signal beam into state $|L\rangle_s$. If we do this we find that our purities and fidelities are improved. We have repeated the state measurements described above (analogous to those presented in Fig. 3) with this polarizer in place and find that for all three measured states $\mathrm{Tr}(\hat{\rho}^2) \geq 0.998$ and $F \geq 0.998$. These states are extremely pure and are well described by the state $|L\rangle_s$.

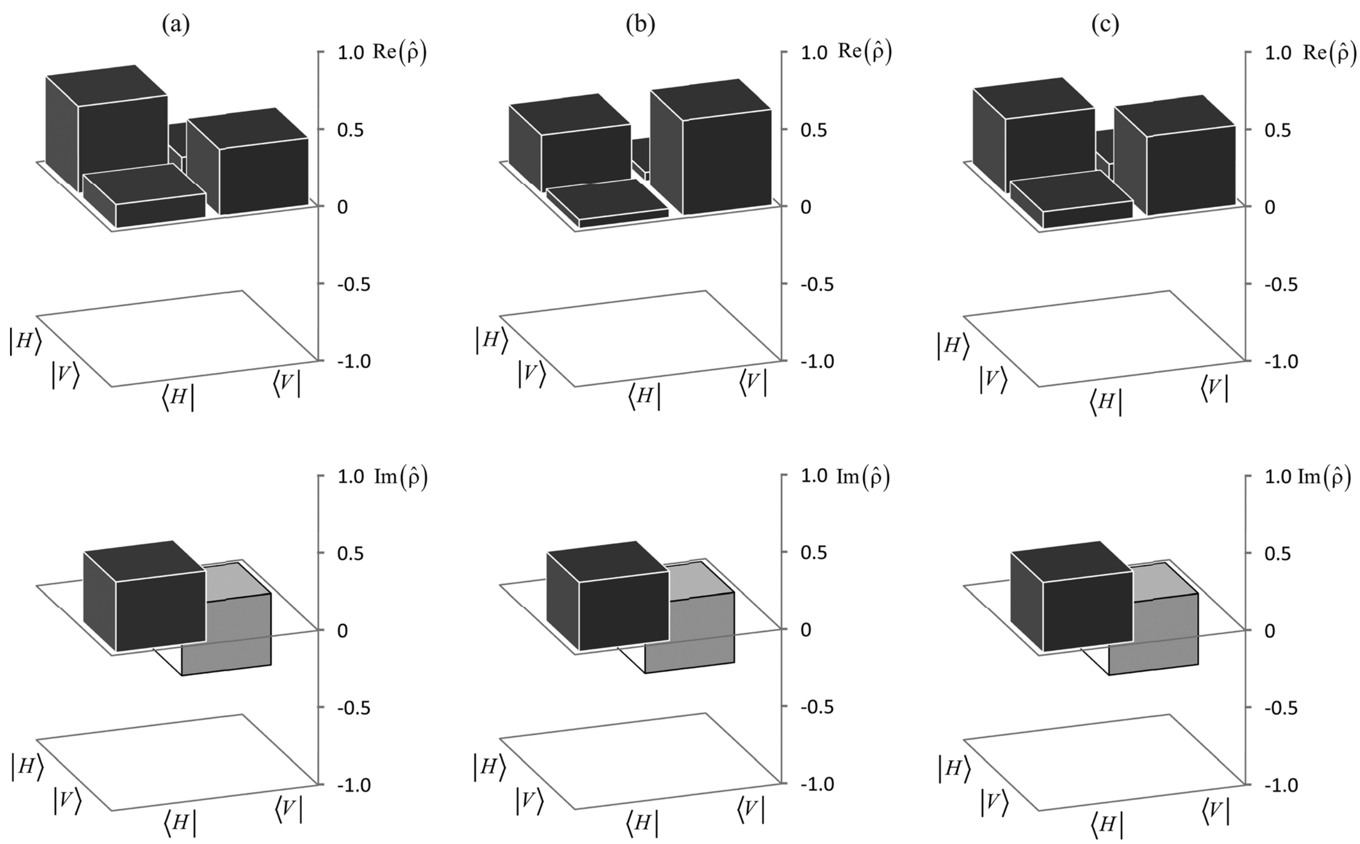

Fig. 3. The real and imaginary parts of the density matrix for the signal beam prepared in state $|L\rangle_s$. The measurements were conditioned on idler photons detected to be: (a) left-circularly polarized at A, (b) right-circularly polarized at A', (c) present at either A or A'. Dark boxes correspond to positive values while light boxes correspond to negative values.

C. Entangled state

Now we adjust the source (as described in Refs. 1, 9, and 10) to produce photons in the state $|\phi^+\rangle$ of Eq. (6). We insert a quarter-wave plate into the idler beam, so that left-circularly polarized photons are detected at A and right-circularly polarized photons are detected at A'. We do not insert a polarizer or wave plate into the signal beam to prepare it in any particular state. Instead, we simply measure the state of the signal photon, conditioned on an idler photon detection at A and/or A'.

The measured state of the signal beam is shown in Fig. 4. In Fig. 4(a), the state of the signal photons is found to closely approximate the right-circular state $\hat{\rho}_R$ ($F = 0.87$), conditioned upon the idler photons having left-circular polarization (the state of the signal is the complex conjugate of that of the idler). This is what we would expect because, as discussed in Sec. II C and the Appendix, for source photons in state $|\phi^+\rangle$, if the idler photon is projected onto the elliptical polarization state $|e\rangle_i$ the signal photon is projected into the complex-conjugate state $|e^*\rangle_s$. Figure 4(b) confirms this behavior. There we see that if the idler photon is measured to be right-circularly polarized, the signal photon is in a state that is well described by the left-circular polarization state $\hat{\rho}_L$ ($F = 0.93$).

In Fig. 4(c), the signal beam state measurement is conditioned on the detection of photons of either polarization in the idler beam. The measured state is found to be mixed—$\mathrm{Tr}(\hat{\rho}^2) = 0.52$—and to closely resemble the state

$$\hat{\rho}_s = \frac{1}{2}\left(|H\rangle_{ss}\langle H| + |V\rangle_{ss}\langle V|\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{37}$$

which corresponds to the signal photon being horizontally polarized half of the time and vertically polarized the other half of the time. As described in the Appendix, this is the result we expect for this measurement.

It is interesting to note that the data used to determine the states displayed in Fig. 4 were all measured in the same experiment at the same time, but that we are able to "measure" three different states. The three states are all *conditioned* on different detection events in the idler beam, and such conditioning (or "post-selecting") is done by sorting the data after it has been acquired. Note that we would have measured essentially the same result as that depicted in Fig. 4(c) if we had removed the polarizer from the idler beam and conditioned the measurements on the detection of a photon in this beam.

The fact that measurements performed in one place affect the results of measurements performed in another place (as depicted in Fig. 4) is a consequence of the nonlocal character of the entangled state $|\phi^+\rangle$. However, the results shown in Fig. 4 are *not* sufficient to prove that the source produces photons in an entangled state. These results would be essentially identical if the source produced photons in the classical mixed state

$$\hat{\rho}_{\mathrm{mix}} = \frac{1}{2}\left(|L,R\rangle\langle L,R| + |R,L\rangle\langle R,L|\right), \tag{38}$$

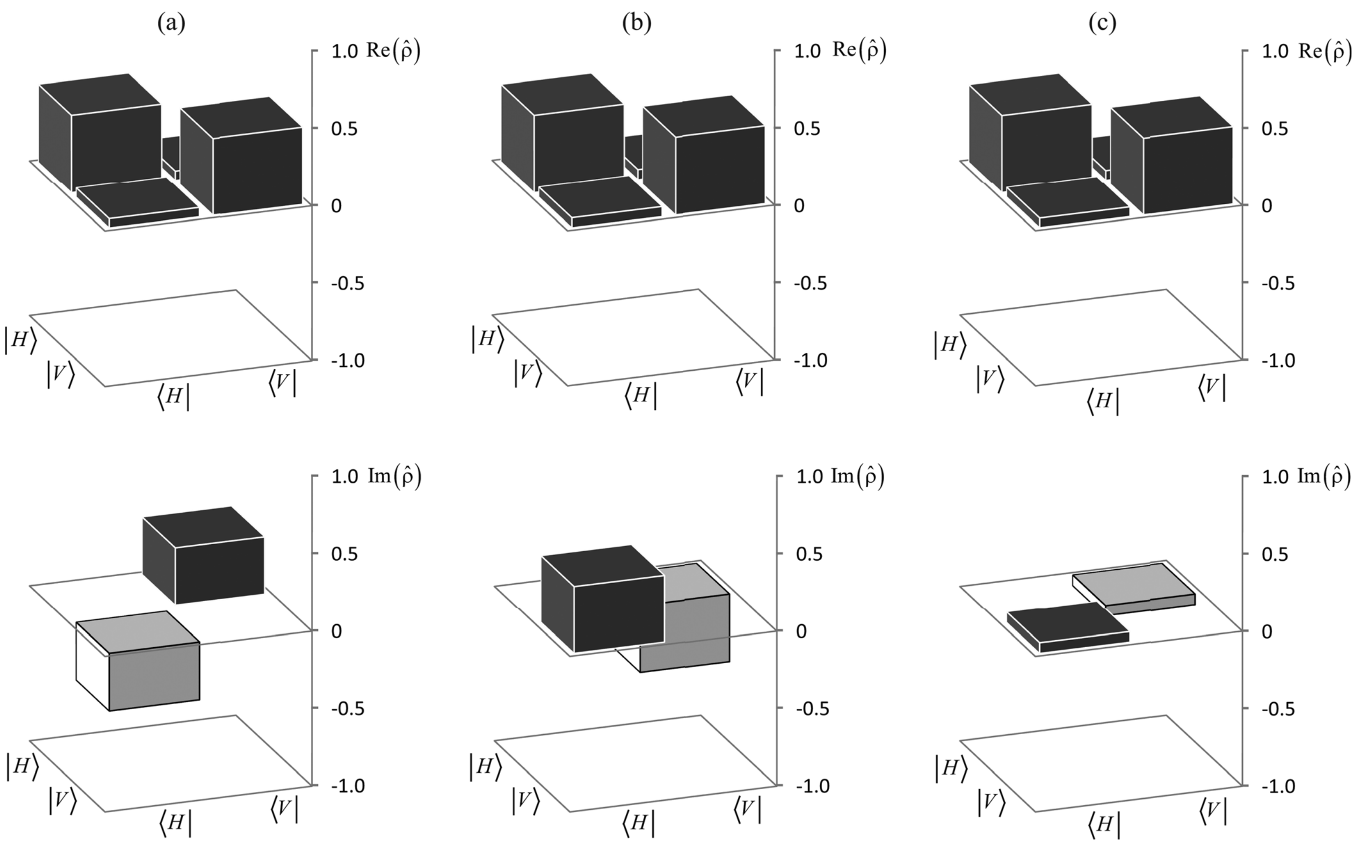

Fig. 4. The real and imaginary parts of the density matrix for the signal beam, with source photons prepared in the entangled state $|\phi^+\rangle$. The measurements were conditioned on idler photons detected to be: (a) left-circularly polarized at A, (b) right-circularly polarized at A', (c) present at either A or A'. Dark boxes correspond to positive values, while light boxes correspond to negative values.

which corresponds to the photons being prepared in the state $|L,R\rangle$ half of the time, and the state $|R,L\rangle$ the other half of the time. In any *single* basis, it is possible for a classical mixed state to have perfect correlations between measurements performed in two locations, and thus mimic the behavior of an entangled state (e.g., the state of Eq. (38) can mimic the behavior of $|\phi^+\rangle$ for measurements in the circular polarization basis). However, this is *only* true in a single basis. If measurements are performed in a different basis the classical correlations will no longer be perfect, whereas the quantum correlations of an entangled state persist for measurements in any basis.

To verify that we are seeing true quantum correlations, we have performed measurements in other bases as well. In Fig. 5(a), we show the measured state of the signal photon conditioned upon the measurement of an idler photon being horizontally polarized. We see that the state is well described by $\hat{\rho}_H$ ($F = 0.97$), which is what we would expect for photons prepared in the entangled state $|\phi^+\rangle$ (as described in Sec. II C). If instead the photons had been prepared in the state $\hat{\rho}_{\text{mix}}$ of Eq. (38), the measured state would have been given by the mixed state of Eq. (37); this can be seen by using Eq. (22) and inserting $\hat{\rho}_{\text{mix}}$ for $\hat{\rho}$ and $|H\rangle_i$ for $|a\rangle_1$.

In Fig. 5(b), we show the measured state of the signal photon conditioned upon the measurement of an idler photon being linearly polarized along –45°. The state is well described by $\hat{\rho}_{-45}$ ($F = 0.89$), which is what we would expect for photons prepared in the entangled state $|\phi^+\rangle$. Once again, photons prepared in $\hat{\rho}_{\text{mix}}$ would have yielded a signal state given by Eq. (37) for this measurement.

The mechanisms described at the end of Sec. III B also contribute to the lack of purity of the entangled state $|\phi^+\rangle$. Additionally, there are other factors that can degrade the purity of an entangled state. For the entangled state to be pure, the photons produced in the two crystals must be completely indistinguishable. Any information that might *in principle* allow one to determine the polarization of a photon would collapse its polarization state and destroy the entanglement. For example, if one was able to separately image the two down-conversion crystals and determine which crystal a photon was produced in, one would know the photon polarization and the entanglement would be destroyed. Partial information reduces the purity of an entangled state, and hence the fidelity of the measurements, without completely destroying the entanglement. In practice, it is difficult to produce photons that are absolutely indistinguishable.

To summarize our measurements for the entangled state, we have measured the state of the signal beam, conditioned on projective measurements performed on the idler beam. These measurements, shown in Figs. 4 and 5, display correlations between the signal and idler beams that are consistent with an entangled state, but not with a classical mixed state. It is possible to imagine a classical state that could fully describe the results of Fig. 4 and Eq. (38), but this state could

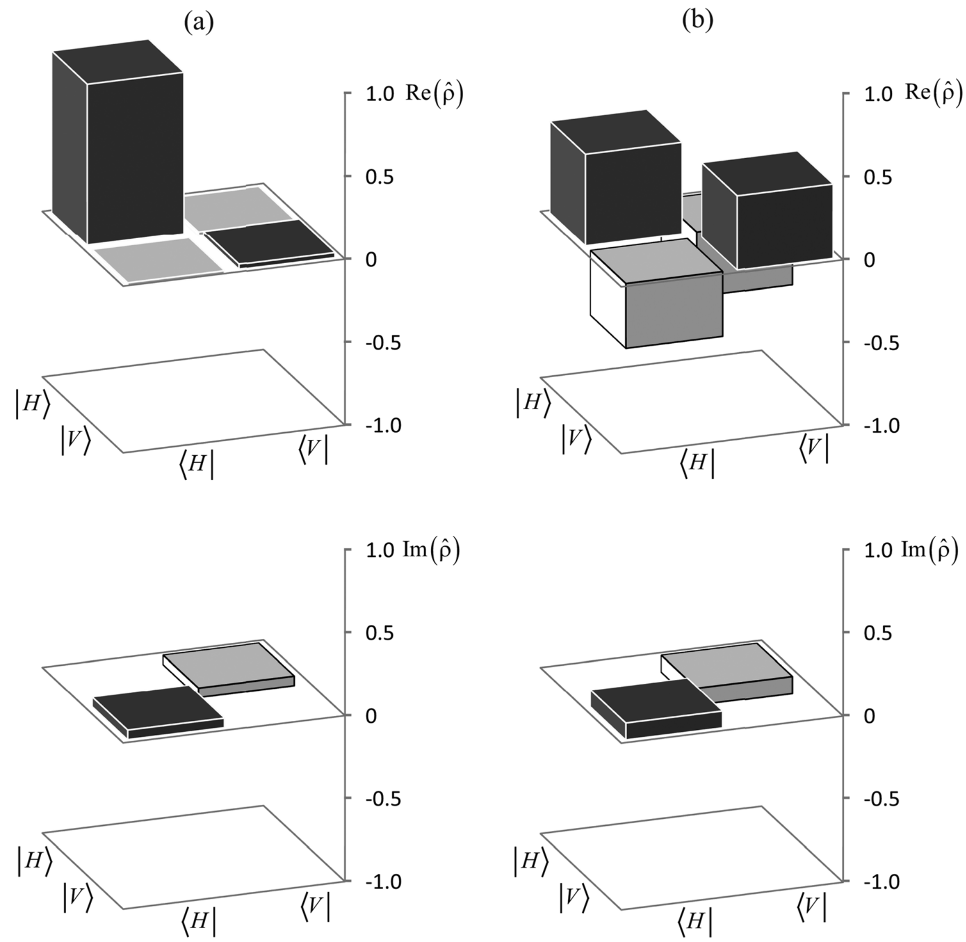

Fig. 5. The real and imaginary parts of the density matrix for the signal beam, with source photons prepared in the entangled state $|\phi^+\rangle$. The measurements were conditioned on idler photons detected to be: (a) horizontally polarized at A, (b) linearly polarized along $-45°$ at A. Dark boxes correspond to positive values while light boxes correspond to negative values.

not also explain the results shown in Fig. 5. This classical state would yield completely mixed states for measurements corresponding to those presented in Fig. 5.[17] These results demonstrate that the correlations we observe cannot be explained classically and are due to the nonlocal character of the entangled state $|\phi^+\rangle$.

IV. CONCLUSIONS

We have performed quantum state measurements of one photon of a two-photon pair produced by spontaneous parametric down-conversion. This state measurement is conditioned on the results of measurements performed on the other photon. We have also presented theoretical results that allow us to describe how the measurements performed on the second photon will affect the state of the first photon.

When the two photons are produced in a non-entangled state, the measured state of the first photon is independent of measurements performed on the second photon (we always measure the same state for the first photon). This is demonstrated in Fig. 3 by observing that all three measured states are similar. However, when the photons are produced in an entangled state, the measured state of the first photon does depend on the results of measurements performed on the second photon. This is demonstrated in Fig. 4 by observing that the three measured states are very different. Measurements performed on the second photon change the state of the two-photon system, which projects the first photon into different states.

Note that the state of the first photon depends on measurements performed on the second, no matter what basis is used for the measurements. Since strong classical correlations should exist only for measurements performed in one basis, we conclude that our photons are prepared in an entangled state. It is the nonlocal nature of entanglement that allows the results of measurements performed in one place to depend on the results of measurements performed somewhere else.

ACKNOWLEDGMENTS

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APPENDIX

Assume that a system is prepared in state $\hat{\rho}$ and that a projective measurement is performed on the system. The state of the system after the measurement $\hat{\rho}'$ is given by[18]

$$\hat{\rho}' = \frac{\hat{P}\hat{\rho}\hat{P}}{\mathrm{Tr}(\hat{P}\hat{\rho}\hat{P})}, \tag{A1}$$

where $\hat{P}$ is an operator that projects onto the state corresponding to the results of the measurement.

We are interested a two-particle system, and we wish to determine the state of particle 2 after a projective measurement is performed on particle 1. Thus, the projection operator operates in the subspace of particle 1 and can be written as

$$\hat{P}_1 = \left(\sum_n |\psi_n\rangle_{1\,1}\langle\psi_n|\right), \tag{A2}$$

where the states $|\psi_n\rangle_1$ are eigenstates of the observable corresponding to the measurement. These states are orthogonal, but they need not form a complete set of states for particle 1. The sum in Eq. (A2) is over the states corresponding to the possible measurement results recorded by the detector. Note that if we square Eq. (A2) we obtain

$$\hat{P}_1^2 = \left(\sum_n |\psi_n\rangle_{1\,1}\langle\psi_n|\right)\left(\sum_m |\psi_m\rangle_{1\,1}\langle\psi_m|\right) = \sum_n\sum_m \left(|\psi_n\rangle_{1\,1}\langle\psi_n|\psi_m\rangle_{1\,1}\langle\psi_m|\right), \tag{A3}$$

which simplifies to

$$\hat{P}_1^2 = \sum_n |\psi_n\rangle_{1\,1}\langle\psi_n| = \hat{P}_1. \tag{A4}$$

Thus, the square of a projection operator is equal to the original projection operator.

Now, we will assume that the states $|\alpha_m\rangle_1$ form a basis for particle 1, the states $|\beta_n\rangle_2$ form a basis for particle 2, and the states $|\alpha_m, \beta_n\rangle$ form a basis for the two-particle system. We wish to determine the state of particle 2, $\hat{\rho}_2$, if the state of the two-particle system is given by $\hat{\rho}'$ in Eq. (A1). The state $\hat{\rho}_2$ is sometimes referred to as the partial density operator, or the reduced density operator, and it is obtained from $\hat{\rho}'$ by performing a partial trace over the state space of particle 1:[14]

$$\hat{\rho}_2 = \mathrm{Tr}_1(\hat{\rho}') = \sum_m {}_1\langle\alpha_m|\hat{\rho}'|\alpha_m\rangle_1. \tag{A5}$$

The matrix elements of $\hat{\rho}_2$ are given by

$${}_2\langle\beta_n|\hat{\rho}_2|\beta_{n'}\rangle_2 = \sum_m \langle\alpha_m, \beta_n|\hat{\rho}'|\alpha_m, \beta_{n'}\rangle. \tag{A6}$$

Note that the denominator in Eq. (A1) is present to normalize the density operator, so for the moment we will concern ourselves only with the numerator and will normalize our results at the end. The partial trace of the numerator in Eq. (A1) is given by

$$\mathrm{Tr}_1(\hat{P}_1\hat{\rho}\hat{P}_1) = \sum_m {}_1\langle\alpha_m|\hat{P}_1\hat{\rho}\hat{P}_1|\alpha_m\rangle_1 = \sum_m {}_1\langle\alpha_m|\hat{P}_1\hat{1}_1\hat{\rho}\hat{P}_1|\alpha_m\rangle_1. \tag{A7}$$

If we express the identity operator for particle 1, $\hat{1}_1$, as a sum over projectors onto a complete set of states, we can write this as

$$\mathrm{Tr}_1(\hat{P}_1\hat{\rho}\hat{P}_1) = \sum_m\sum_i {}_1\langle\alpha_m|\hat{P}_1|\alpha_i\rangle_{1\,1}\langle\alpha_i|\hat{\rho}\hat{P}_1|\alpha_m\rangle_1. \tag{A8}$$

We now rearrange the terms in the sum, which yields

$$\mathrm{Tr}_1(\hat{P}_1\hat{\rho}\hat{P}_1) = \sum_m\sum_i {}_1\langle\alpha_i|\hat{\rho}\hat{P}_1|\alpha_m\rangle_{1\,1}\langle\alpha_m|\hat{P}_1|\alpha_i\rangle_1 = \sum_i {}_1\langle\alpha_i|\hat{\rho}\hat{P}_1\hat{1}_1\hat{P}_1|\alpha_i\rangle_1, \tag{A9}$$

and further simplifies to

$$\mathrm{Tr}_1\left(\hat{P}_1\hat{\rho}\hat{P}_1\right) = \sum_i {}_1\langle\alpha_i|\hat{\rho}\hat{P}_1{}^2|\alpha_i\rangle_1 = \sum_i {}_1\langle\alpha_i|\hat{\rho}\hat{P}_1|\alpha_i\rangle_1. \tag{A10}$$

To obtain the final expression in this equation, we have used Eq. (A4).

We will now consider two special cases. In the first we assume that $\hat{P}_1$ projects onto a single state $|a\rangle_1$ so that $\hat{P}_1 = |a\rangle_{1\,1}\langle a|$. In this case, Eq. (A10) becomes

$$\begin{aligned}\mathrm{Tr}_1\left(\hat{P}_1\hat{\rho}\hat{P}_1\right) &= \sum_i {}_1\langle\alpha_i|\hat{\rho}|a\rangle_{1\,1}\langle a|\alpha_i\rangle_1 \\ &= \sum_i {}_1\langle a|\alpha_i\rangle_{1\,1}\langle\alpha_i|\hat{\rho}|a\rangle_1,\end{aligned} \tag{A11}$$

which simplifies to

$$\mathrm{Tr}_1\left(\hat{P}_1\hat{\rho}\hat{P}_1\right) = {}_1\langle a|\hat{1}_1\hat{\rho}|a\rangle_1 = {}_1\langle a|\hat{\rho}|a\rangle_1. \tag{A12}$$

We can now normalize this expression to obtain

$$\hat{\rho}_2 = \frac{{}_1\langle a|\hat{\rho}|a\rangle_1}{\mathrm{Tr}\left({}_1\langle a|\hat{\rho}|a\rangle_1\right)}. \tag{A13}$$

This is the result given in Eq. (22).

As an example, consider a source that produces photons in the entangled polarization state $|\phi^+\rangle$ whose density operator is given in Eq. (24). A polarization measurement is performed on the idler photon and it is found to be elliptically polarized with a corresponding polarization state $|e\rangle_i$ given by Eq. (1). To calculate the state of the signal photon after this measurement, we begin by calculating the numerator of Eq. (A13) and find

$$\begin{aligned}{}_i\langle e|\hat{\rho}|e\rangle_i = \frac{1}{2}(a\,{}_i\langle H| + be^{-i\phi}\,{}_i\langle V|)(|H,H\rangle\langle H,H| \\ + |H,H\rangle\langle V,V| + |V,V\rangle\langle H,H| \\ + |V,V\rangle\langle V,V|)(a|H\rangle_i + be^{i\phi}|V\rangle_i).\end{aligned} \tag{A14}$$

Expanding this, we see that

$$\begin{aligned}{}_i\langle e|\hat{\rho}|e\rangle_i = \frac{1}{2}(a|H\rangle_s\langle H,H| + a|H\rangle_s\langle V,V| \\ + be^{-i\phi}|V\rangle_s\langle H,H| + be^{-i\phi}|V\rangle_s\langle V,V|) \\ \times\left(a|H\rangle_i + be^{i\phi}|V\rangle_i\right),\end{aligned} \tag{A15}$$

which simplifies to

$$\begin{aligned}{}_i\langle e|\hat{\rho}|e\rangle_i = \frac{1}{2}\Big(a^2|H\rangle_{ss}\langle H| + abe^{i\phi}|H\rangle_{ss}\langle V| \\ + abe^{-i\phi}|V\rangle_{ss}\langle H| + b^2|V\rangle_{ss}\langle V|\Big).\end{aligned} \tag{A16}$$

Summing the coefficients of the diagonal terms, we find that the trace of this operator is

$$\mathrm{Tr}\left({}_i\langle e|\hat{\rho}|e\rangle_i\right) = \frac{1}{2}(a^2 + b^2) = \frac{1}{2}. \tag{A17}$$

The density operator of the signal beam photon is thus

$$\begin{aligned}\hat{\rho}_s = (a^2|H\rangle_{ss}\langle H| + abe^{i\phi}|H\rangle_{ss}\langle V| + abe^{-i\phi}|V\rangle_{ss}\langle H| \\ + b^2|V\rangle_{ss}\langle V|).\end{aligned} \tag{A18}$$

This can be written more simply as

$$\begin{aligned}\hat{\rho}_s &= (a|H\rangle_s + be^{-i\phi}|V\rangle_s)(a\,{}_s\langle H| + be^{i\phi}\,{}_s\langle V|) \\ &= |e^*\rangle_{ss}\langle e^*|,\end{aligned} \tag{A19}$$

and we see that the measurement on the idler photon projects the signal photon into the complex conjugate state $|e^*\rangle_s$ of Eq. (27).

The second case we are interested in is one in which we simply register the presence of particle 1. We are effectively projecting onto all possible states, and hence the projection operator is the identity operator:

$$\hat{P}_1 = \hat{1}_1. \tag{A20}$$

In this case, Eq. (A10) reduces to the partial trace of $\hat{\rho}$:

$$\mathrm{Tr}_1\left(\hat{P}_1\hat{\rho}\hat{P}_1\right) = \mathrm{Tr}_1(\hat{\rho}) = \sum_i {}_1\langle\alpha_i|\hat{\rho}|\alpha_i\rangle_1. \tag{A21}$$

To find $\hat{\rho}_2$, we normalize this result by tracing it over the state space of particle 2:

$$\mathrm{Tr}_2[\mathrm{Tr}_1(\hat{\rho})] = \sum_{i,j} {}_1\langle\alpha_i,\beta_j|\hat{\rho}|\alpha_i,\beta_j\rangle_1 = \mathrm{Tr}(\hat{\rho}) = 1, \tag{A23}$$

which means that it is already normalized. In this case, the state of particles 2 is simply given by

$$\hat{\rho}_2 = \mathrm{Tr}_1(\hat{\rho}). \tag{A24}$$

Once again, consider the example of photons in the entangled polarization state $|\phi^+\rangle$, whose density operator is given by Eq. (24). If we simply detect the presence of an idler photon, the signal photon is projected onto the state

$$\hat{\rho}_s = \mathrm{Tr}_i(\hat{\rho}) = {}_i\langle H|\hat{\rho}|H\rangle_i + {}_i\langle V|\hat{\rho}|V\rangle_i. \tag{A25}$$

Expanding, we find that

$$\begin{aligned}\hat{\rho}_s = \frac{1}{2}[{}_i\langle H|(|H,H\rangle\langle H,H| + |H,H\rangle\langle V,V| \\ + |V,V\rangle\langle H,H| + |V,V\rangle\langle V,V|)|H\rangle_i \\ + {}_i\langle V|(|H,H\rangle\langle H,H| + |H,H\rangle\langle V,V| \\ + |V,V\rangle\langle H,H| + |V,V\rangle\langle V,V|)|V\rangle_i],\end{aligned} \tag{A26}$$

which then simplifies to

$$\hat{\rho}_s = \frac{1}{2}\left(|H\rangle_{ss}\langle H| + |V\rangle_{ss}\langle V|\right). \tag{A27}$$

Physically, this represents a classical mixed state in which the polarization of the signal photon is random; half of the time a photon will be found to be horizontally polarized and half of the time it will be found to be vertically polarized.

Finally, we note that the randomness of the polarization for a photon in the state of Eq. (A27) is not limited to measurements performed in the horizontal-vertical basis. For this state, the polarization will be found to be random for measurements performed in any basis.

[a]Electronic mail: beckmk@whitman.edu

[1]M. Beck, *Quantum Mechanics: Theory and Experiment* (Oxford U.P., Oxford, 2012).

[2]M. G. Raymer, "Measuring the quantum mechanical wave function," Contemp. Phys. **38**, 343–355 (1997).

[3]Ulf Leonhardt, *Measuring the Quantum State of Light* (Cambridge U.P., Cambridge, UK, 1997).

[4]*Quantum State Estimation*, edited by M. G. A. Paris and J. Rehacek (Springer-Verlag, Berlin, 2004).

[5]J. B. Altepeter, E. R. Jeffrey, and P. G. Kwiat, "Photonic state tomography," in *Advances in Atomic, Molecular and Optical Physics*, edited by P. R. Berman and C. C. Lin (Elsevier, Amsterdam, 2006), Vol. 52, pp. 105–159.

[6]D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani, "Measurement of the Wigner distribution and the density matrix of a light mode using optical homodyne tomography: Application to squeezed states and the vacuum," Phys. Rev. Lett. **70**, 1244–1247 (1993).

[7]M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge U.P., Cambridge, 2000).

[8]D. Dehlinger and M. W. Mitchell, "Entangled photon apparatus for the undergraduate laboratory," Am. J. Phys. **70**, 898–902 (2002).

[9]D. Dehlinger and M. W. Mitchell, "Entangled photons, nonlocality, and Bell inequalities in the undergraduate laboratory," Am. J. Phys. **70**, 903–910 (2002).

[10]J. A. Carlson, M. D. Olmstead, and M. Beck, "Quantum mysteries tested: An experiment implementing Hardy's test of local realism," Am. J. Phys. **74**, 180–186 (2006).

[11]E. J. Galvez, "Qubit quantum mechanics with correlated-photon experiments," Am. J. Phys. **78**, 510–519 (2010).

[12]J. Carvioto-Lagos, G. Armendariz, V. Velazquez, E. Lopez-Moreno, M. Grether, and E. J. Galvez, "The Hong-Ou-Mandel interferometer in the undergraduate laboratory," Eur. J. Phys. **33**, 1843–1850 (2012).

[13]Letter from Albert Einstein to Max Born, 3 March 1947, *The Born-Einstein Letters; Correspondence between Albert Einstein and Max and Hedwig Born from 1916 to 1955*, Translated by Irene Born (Walker, New York, 1971).

[14]C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics* (John Wiley and Sons, New York, 1977), pp. 295–307.

[15]E. Hecht, *Optics*, 4th ed. (Addison-Wesley, San Francisco, 2002), pp. 374–375.

[16]Modern Quantum Mechanics Experiments for Undergraduates, <http://www.whitman.edu/~beckmk/QM>.

[17]Indeed, this is why we have chosen to perform measurements in the bases that we have (horizontal-vertical, ±45°, and left- right-circular). Classically, perfect correlations in one of these bases lead to no correlation in the others; for a quantum mechanical entangled state the correlations are maintained.

[18]Leslie E. Ballentine, *Quantum Mechanics: A Modern Development* (World Scientific, Singapore, 1998), pp. 244–247.

Demonstration Multimeter

This demonstration meter, about 20 cm high, has all the functions of the once popular Simpson Meter except the portability. It probably dates from about 1950, based on the use of the 1N34 crystal diode rectifier. The basic galvanometer has a full-scale-deflection rating of 500 μA, which makes it one tenth as sensitive as the Simpson. It has no maker's name and is on long-term loan to the Greenslade Collection from Appalachian State University. (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College)