EXPLICIT FORMULAS FOR 2-CHARACTERS

ANGÉLICA M. OSORNO

Abstract. Ganter and Kapranov associated a 2-character to 2-representations of a finite group. Elgueta classified 2-representations in the category of 2-vector spaces $2Vect_k$ in terms of cohomological data. We give an explicit formula for the 2-character in terms of this cohomological data and derive some consequences.

1. Introduction

In [6], Hopkins, Kuhn and Ravenel develop a theory of generalized characters that computes $E^*(BG)$ for the n-th Morava E-theory. The characters in this case are class functions defined on the set of n-tuples of commuting elements of G whose order is a power of p. In [5], Ganter and Kapranov define a 2-character for a 2-representation of a finite group in a 2-category. This 2-character is a function that assigns an element of the field k to every pair (g, h) of commuting elements in G. Ganter and Kapranov proved that these 2-characters satisfy the same formulas as the characters in [6] for $n = 2$.

The purpose of this paper is to find an explicit description of the 2-characters of a 2-representation in the 2-category of 2-vector spaces, $2Vect_k$. In order to find this description, we first review the algebraic classification of 2-representations. In [3], it is shown that every equivalence class of 2-representations is given uniquely by a finite G-set S and a class in $H^2(G; k^S)$. We present a streamlined approach to this result.

We then proceed to compute the 2-character in terms of this associated cohomology class. Using these computations we prove that 2-characters are additive and multiplicative with respect to direct sum and tensor product of 2-representations.

Given a 2-representation ρ of $H \subseteq G$, Ganter and Kapranov also define the induced representation, and compute its character in terms of the character of ρ. Using our cohomological classification of representations, we identify the induced representation in terms of the cohomological data for ρ using the Shapiro isomorphism.

Finally, using the explicit computation of 2-characters we give an example of two non-equivalent 2-representations that have the same character, thus showing that this assignment is not faithful.

In [1] and [2], Bartlett develops independently of [5] a theory of categorical characters for 2-representations, although he works with 2-Hilbert spaces as opposed to 2-vector spaces. Bartlett defines the 2-character of a 2-representation, which is the same as the categorical character of [5], and proves that it gives an equivariant vector bundle over the group.

In [5], Ganter and Kapranov use this action of the group on the bundle to define the 2-character, which is a discrete invariant of the 2-representation. In this paper, we work directly to compute this discrete invariant in terms of the cohomological data. An alternative approach to the results of this paper is to translate Bartlett's results into the cohomological language.

The 2-categorical language can be cumbersome, so we review some of the terminology and constructions. By 2-category, 2-functor and 2-natural transformation we mean the weak versions, that is, what other authors call bicategories, pseudofunctors and pseudonatural transformations. For more background on 2-categories we refer the reader to [5] and [3].

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2. 2-representations and their characters

Following [5], we review the notions of 2-representations of a group and character theory.

Definition 1. Let $\mathcal{C}$ be a 2-category and G a group. A *2-representation* of G in $\mathcal{C}$ is a 2-functor from G (viewed as a discrete 2-category) to $\mathcal{C}$.

This amounts to the following data:

(1) an object V of $\mathcal{C}$,
(2) for every $g \in G$, a 1-morphism $\rho_g : V \to V$,
(3) a 2-isomorphism $\phi_1 : \rho_1 \Rightarrow \mathbf{1}_V$,
(4) for every pair $g, h \in G$, a 2-isomorphism $\phi_{g,h} : \rho_g \circ \rho_h \Rightarrow \rho_{gh}$.
    This data has to satisfy the following conditions:
(5) (associativity) for every $g, h, k \in G$,

$$\begin{tikzcd}
(\rho_g \circ \rho_h) \circ \rho_k \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\phi_{g,h} \circ \rho_k"'] & \rho_g \circ (\rho_h \circ \rho_k) \arrow[r, Rightarrow, "\rho_g \circ \phi_{h,k}"] & \rho_g \circ \rho_{hk} \arrow[d, Rightarrow, "\phi_{g,hk}"] \\
\rho_{gh} \circ \rho_k \arrow[rr, Rightarrow, "\phi_{gh,k}"'] & & \rho_{ghk}
\end{tikzcd}$$

must commute,
(6) for any $g \in G$,

$$\begin{tikzcd}
\rho_1 \circ \rho_g \arrow[rr, Rightarrow, "\phi_1 \circ \rho_g"] \arrow[dr, Rightarrow, "\phi_{1,g}"'] & & \mathbf{1}_V \circ \rho_g \arrow[dl, Rightarrow, "\epsilon"] \\
& \rho_g &
\end{tikzcd} , \quad
\begin{tikzcd}
\rho_g \circ \rho_1 \arrow[rr, Rightarrow, "\rho_g \circ \phi_1"] \arrow[dr, Rightarrow, "\phi_{g,1}"'] & & \rho_g \circ \mathbf{1}_V \arrow[dl, Rightarrow, "\zeta"] \\
& \rho_g &
\end{tikzcd}$$

must commute.
    Here α, ϵ and ζ are the associativity and left and right unit 2-isomorphisms of $\mathcal{C}$, as in [5, 2.1].

There is a 2-category in which we are particularly interested: the 2-category of 2-vector spaces. There are several 2-categories which are 2-equivalent and give equivalent 2-representation theories. We will use the definition in [7].

Definition 2. Let k be a field. The 2-category $2Vect_k$ has as objects $[n]$, where $n \in \{0, 1, 2, \dots\}$. For integers m, n, the set of 1-morphisms $1Hom_{2Vect_k}([m], [n])$ is the set of $m \times n$ matrices with entries in k-vector spaces. These are called 2-matrices. Composition is given by matrix multiplication using tensor product and direct sum. For 2-matrices A and B of the same size, 2-morphisms are given by matrices of linear maps ϕ, with $\phi_{ij} : A_{ij} \to B_{ij}$.

Remark 3. This 2-category is not strict. To make sense of the matrix multiplication of $A \in 1Hom_{2Vect_k}([m],[n])$ and $B \in 1Hom_{2Vect_k}([n],[p])$, one has to choose a parenthesization of the direct sum of the n terms

$$(A_{i,j} \otimes B_{j,k})_{j=1}^{n}.$$

One notices then that $A \cdot (B \cdot C)$ and $(A \cdot B) \cdot C$ are not equal but naturally isomorphic. In [4], there is a construction of a strict version which is equivalent to $2Vect_k$.

Thus a 2-representation of a group G in $2Vect_k$ consists of the following data:

(1) A natural number n, called the dimension,
(2) for every $g \in G$, an $n \times n$ 2-matrix, ρ_g
(3) a 2-isomorphism $\phi_1 : \rho_1 \Rightarrow \mathbf{1}_{[n]}$,
(4) for every pair $g, h \in G$, a 2-isomorphism $\phi_{g,h} : \rho_g \circ \rho_h \Rightarrow \rho_{gh}$.

The isomorphisms $\phi_{g,h}$ and ϕ_1 are subject to the same conditions expressed above.

The following result is similar to a more general one in [3]. The result here is presented in a coordinate-free approach. We will later use this result to recover [5, Prop. 7.3] in Proposition 7.

Proposition 4. *There is a one-to-one correspondence between equivalence classes of 2-representations of G in $2Vect_{\mathbb{C}}$ and pairs $(S,[c])$ where S is a finite G-set and $[c] \in H^2(G;(\mathbb{C}^\times)^S)$. Here $(\mathbb{C}^\times)^S$ denotes $(\mathbb{C}^\times)^{|S|}$ as a G-module through the action of G on S.*

Proof. Note that since $\rho_g\rho_{g^{-1}}$ is isomorphic to $\mathbf{1}_{[n]}$, each ρ_g is given by a weakly invertible 2-matrix. This means that the entries in ρ_g can only be 0 and 1-dimensional vector spaces, with exactly one entry per row and column being 1-dimensional. That is, up to isomorphism, ρ_g is given by an $n \times n$ permutation matrix. Thus, we can think of ρ as a map

$$\rho : G \to \Sigma_n.$$

Now, let us turn our attention to $\phi_{g,h}$ and ϕ_1. The 2-matrices ρ_{gh} and $\rho_g\rho_h$ have only one nonzero entry per row and column, and those entries are 1-dimensional vector spaces. Thus, to specify the 2-isomorphism $\phi_{g,h}$ all we need to give is a sequence of n nonzero complex numbers $\{c_i(g,h)\}$ which give the isomorphism for the nonzero entry in the ith row.

Condition (5) in the definition of a 2-representation implies

$$c_{\sigma^{-1}(i)}(h,k) \cdot c_i(g,hk) = c_i(gh,k) \cdot c_i(g,h),$$

where σ is the permutation represented by ρ_g.

We can think of $(\mathbb{C}^\times)^n$ as a G-module through ρ, where $g \cdot \overrightarrow{a} = \rho_g \overrightarrow{a}$ in matrix notation. We will denote this G-module by $(\mathbb{C}^\times)^n_\rho$.

We can then think of c as a 2-cochain $G \times G \to (\mathbb{C}^\times)^n_\rho$. Then the condition above becomes the cocycle condition:

$$(\delta c)(g,h,k) = g \cdot c(h,k) - c(gh,k) + c(g,hk) - c(g,h) = 0$$

Here we are using additive notation for the component-wise multiplication group structure of $\mathbb{C}^\times)^n$.

On the other hand, Condition (6) of Definition 1 with $g = 1$ implies that ϕ_1 is given by multiplication by $c(1,1)$.

Hence, we can say that up to isomorphism, a 2-representation is determined by a group homomorphism $\rho : G \to \Sigma_n$ and a 2-cocycle $c \in C^2(G; (\mathbb{C}^\times)^n_\rho)$. This coincides with the notion in [3].

In this new language, we would like to identify the equivalence classes of representations. Two representations are equivalent if there exists a 2-natural equivalence between the functors. A 2-natural transformation is a 1-morphism $f : [n] \to [n']$ together with a 2-isomorphism $\psi(g) : \rho'_g \circ f \Rightarrow f \circ \rho_g$ for every $g \in G$, satisfying two coherence conditions:

(1) For all $g, h \in G$, $\psi(gh) \cdot (\phi'_{g,h} \circ \mathbf{1}_f) = (\mathbf{1}_f \circ \phi_{g,h}) \cdot (\psi(g) \circ \mathbf{1}_{\rho_h}) \cdot (\mathbf{1}_{\rho'_g} \circ \psi(h))$,
(2) $\phi'_1 \circ \mathbf{1}_f = (\mathbf{1}_f \circ \phi_1) \cdot \psi(1)$.

This 2-natural transformation is an equivalence if and only if f is a weakly invertible 1-morphism, that is if $n = n'$ and f is given by a permutation matrix.

Assume two 2-representations are equivalent. If these 2-representations are given by the same map $\rho : G \to \Sigma_n$ and $f = \mathbf{1}_{[n]}$, the 2-isomorphism $\psi(g)$ is given by a sequence of n nonzero complex numbers $b_i(g)$ which give the isomorphisms on the nonzero 1-dimensional vector spaces in each row. Again, we can think of these vectors of complex numbers as a 1-cochain $G \to (\mathbb{C}^\times)^n$. The two coherence conditions imply for all i:

$$b_i(gh)c'_i(g,h) = c_i(g,h)b_i(g)b_{\sigma^{-1}(i)}(h),$$

where c and c' are the cocycles giving the two representations. If we write this in additive notation we get:

$$(\delta b)(g,h) = g \cdot b(h) - b(gh) + b(g) = c'(g,h) - c(g,h).$$

That is, c and c' are cohomologous cocycles in $C^2(G; (\mathbb{C}^\times)^n_\rho)$ if and only if they give equivalent representations.

In general the representations given by ρ, $[c]$ and ρ', $[c']$ are equivalent if and only if there exists a permutation $f \in \Sigma_n$ such that $\rho'_g = f\rho_g f^{-1}$ and $[c'] = [f \cdot c]$. This follows from the assertions above.

□

3. Direct sum and tensor product

There is a notion of direct sum and tensor product in $2Vect_k$ as noted in [7]. Direct sum is given as follows:

- On objects: $[n] \oplus [m] = [n+m]$,
- on 1-morphisms is given by block sum of 2-matrices,
- on 2-morphisms is given by block sum of matrices of linear maps.

Tensor product is given as follows:

- On objects: $[n] \otimes [n'] = [nn']$.
- on 1-morphisms: let $f : [m] \to [n]$, $f' : [m'] \to [n']$ be 1-morphisms, then $f \otimes f' : [mm'] \to [nn']$ is the 2-matrix with $(i,i'),(j,j')$-entry equal to $f_{ij} \otimes f'_{i'j'}$, where the set of mm' elements is labeled by pairs (i,i'), where $i = 1,\dots,m$, $i' = 1,\dots,m'$, with the order: $(1,1),(1,2),\dots,(1,n'),(2,1),\dots,(m,m')$ and similarly for nn'.
- on 2-morphisms: similarly as above.

These operations can be extended to 2-representations on $2Vect_k$ by taking the appropriate direct sum and/or tensor product of the respective objects, 1-morphisms and 2-morphisms. It is not hard to prove that we obtain a new 2-representation in both cases.

4. Induced 2-Representations

In [5], Ganter and Kapranov define the notion of an induced representation given $H \subseteq G$ and inclusion of finite groups. Here we analyze the case of $2Vect_k$ following their explicit description in Remark 7.2.

Let $\rho : H \to \Sigma_n$, and let $[c] \in H^2(H; (\mathbb{C}^\times)^n_\rho)$ be a 2-representation on $[n]$. Let S be the corresponding H-set. Let m be the index of H in G. Let $\mathcal{R} = \{r_1,\dots,r_m\}$ be a system of representatives of the left cosets of H in G.

Then $\mathrm{ind}\,|_H^G(\rho)$ is a 2-representation of G of dimension nm. The matrix for $\mathrm{ind}\,|_H^G\rho_g$ is a block matrix, with blocks of size $n \times n$, where the (i,j)-th block is given as follows:

$$(\mathrm{ind}\,|_H^G\rho_g)_{ij} = \begin{cases} \rho_h & \text{if} \quad gr_j = r_ih, h \in H \\ 0 & \text{else.} \end{cases}$$

Now we turn our attention to $\mathrm{ind}\,|_H^G\phi_{g_1,g_2}$. Notice that

$$(\mathrm{ind}\,|_H^G\rho_{g_1}) \circ_1 (\mathrm{ind}\,|_H^G\rho_{g_2})_{ik} =$$
$$= \begin{cases} \rho_{h_1} \circ_1 \rho_{h_2} & \text{if} \quad g_1r_j = r_ih_1 \text{ and } g_2r_k = r_jh_2 \\ 0 & \text{else.} \end{cases}$$

and the (i,k)-th block is not zero precisely when $g_1g_2r_k = r_ih_1h_2$, that is, when

$$(\mathrm{ind}\,|_H^G\rho_{(g_1g_2)})_{ik} = \rho_{(h_1h_2)}.$$

Here $\circ_1$ denotes vertical composition of 2-morphisms.

On this block then

$$(\mathrm{ind}\,|_H^G\phi_{g_1,g_2})_{ik} = \phi_{h_1,h_2}.$$

Notice that the G-set given by $\mathrm{ind}\,|_H^G\rho : G \to \Sigma_{nm}$ is precisely

$$\mathrm{ind}\,|_H^G S = G \times_H S \cong \mathcal{R} \times S.$$

Thus, the G-module $(\mathbb{C}^\times)^{\mathrm{ind}\,|_H^G S}$ is

$$\mathrm{coind}\,|_H^G[(\mathbb{C}^\times)^S] = Hom_H(\mathbb{Z}G, (\mathbb{C}^\times)^S).$$

The corresponding cocycle is then

$$(\mathrm{ind}\,|_H^G c)_{(r_i,s)}(g_1, g_2) = c_s(h_1, h_2),$$

where $(r_i, s) \in \mathcal{R} \times S$ and h_1, h_2 are as above.

We shall now recall the Shapiro isomorphism, which is a standard tool in group cohomology. For more information, we refer the reader to [8, 6.3.2].

Theorem 5. *Shapiro Isomorphism. Let G be a group and H a subgroup of G. Let A be an H-module and* $\mathrm{coind}\,|_H^G(A) = Hom_{\mathbb{Z}H}(\mathbb{Z}G, A)$ *the coinduced G-module. Then*

$$H^*(G; \mathrm{coind}\,|_H^G(A)) \cong H^*(H; A).$$

One can trace the class $[c]$ in the isomorphism above to obtain the following result.

Theorem 6. *The cohomology class* $[\mathrm{ind}\,|_H^G c]$ *is the image of* $[c]$ *under the Shapiro isomorphism*

$$H^2(H; (\mathbb{C}^\times)^S) \cong H^2(G; \mathrm{coind}\,|_H^G[(\mathbb{C}^\times)^S]) \cong H^2(G; (\mathbb{C}^\times)^{\mathrm{ind}\,|_H^G S}).$$

In particular, let $(S, [c])$ be an equivalence class of a representation of G. Let

$$S = \coprod_{i=1}^{k} G/H_i$$

be the decomposition of S into G orbits. Then the chain of isomorphisms

$$H^2(G; (\mathbb{C}^\times)^S) \cong \bigoplus_{i=1}^{k} H^2(G; (\mathbb{C}^\times)^{G/H_i}) \cong \bigoplus_{i=1}^{k} H^2(H_i; (\mathbb{C}^\times))$$

sends $[c]$ to $[c_1] \oplus \cdots \oplus [c_k]$ to $[d_1] \oplus \cdots \oplus [d_k]$, where $[d_i]$ is the image of $[c_i]$ under the Shapiro isomorphism.

This means that the representation given by $(G/H_i, [c_i])$ is the induced representation of a 1-dimensional representation $(*, [d_i])$ of H_i

We thus recover the following proposition, which coincides with [5, Prop. 7.3].

Proposition 7. *Every representation is the direct sum of induced 1-dimensional representations.*

5. 2-characters

We would like to know what the character introduced in [5] looks like in terms of ρ and c.

Definition 8. Given a 2-category $\mathcal{C}$, let A be an object and $F : A \to A$ a 1-endomorphism. We define the *categorical trace* as

$$\mathbb{T}r(F) = 2Hom_{\mathcal{C}}(\mathbf{1}_A, F).$$

Note that since $\mathcal{C}$ is a 2-category, $End(A) = Hom(A, A)$ is a category. This definition gives a functor $\mathbb{T}r : End(A) \to Set$. If $\alpha : F \Rightarrow G$ is a 2-morphism between $F, G \in End(A)$,

$$\mathbb{T}r(\alpha) : 2Hom_{\mathcal{C}}(\mathbf{1}_A, F) \to 2Hom_{\mathcal{C}}(\mathbf{1}_A, G)$$

is given by composition with α.

Let $\mathcal{D}$ be a category. We recall that if a 2-category is enriched over $\mathcal{D}$ then the categories $Hom(A, B)$ are enriched over $\mathcal{D}$ for all objects A, B.

Hence, if the 2-category $\mathcal{C}$ is enriched over $\mathcal{D}$ then $\mathbb{T}r$ is a functor into $\mathcal{D}$.

Let $\mathcal{C} = Vect_k$, let $A = [n]$ and let F be an $n \times n$ matrix $[F_{ij}]$ of vector spaces. Then the following equality holds

$$\mathbb{T}r(F) = \bigoplus_{i=1}^{n} F_{ii}.$$

Note that the categorical trace is additive and multiplicative in the following sense. Let $F : [n] \to [n]$ and $G : [m] \to [m]$. Then $F \oplus G : [n+m] \to [n+m]$ and

$$\mathbb{T}r(F \oplus G) = \bigoplus_{i=1}^{n+m} (F \oplus G)_{ii} = (\bigoplus_{i=1}^{n} F_{ii}) \oplus (\bigoplus_{i=1}^{m} G_{ii}) = \mathbb{T}r(F) \oplus \mathbb{T}r(G).$$

Also, $F \otimes G : [n \cdot m] \to [n \cdot m]$ and

$$\begin{aligned} \mathbb{T}r(F \otimes G) &= \bigoplus_{\substack{i=1,\dots,n \\ j=1,\dots,m}} (F \otimes G)_{(i,j)(i,j)} \\ &= \bigoplus_{\substack{i=1,\dots,n \\ j=1,\dots,m}} F_{ii} \otimes G_{jj} \\ &= (\bigoplus_{i=1}^{n} F_{ii}) \otimes (\bigoplus_{i=1}^{m} G_{jj}) \\ &= \mathbb{T}r(F) \otimes \mathbb{T}r(F). \end{aligned}$$

Given $F : A \to A$, let $G : A \to B$ be an equivalence with quasi-inverse H. Then the trace is conjugation invariant in the sense that there is an isomorphism:

$$\psi : \mathbb{T}r(F) \to \mathbb{T}r(GFH).$$

Let ρ be a 2-representation of a group G in $\mathcal{C}$. The categorical character of ρ is given by assigning to each $g \in G$ the trace $\mathbb{T}r(\rho_g)$. In [2] and [1], the author calls this the 2-character, not to be confused with the 2-character defined below.

In this case, since we have the maps $\phi_{g,h}$ and ϕ_1 we can use the conjugation invariance above to get a map

$$\psi_g(h) : \mathbb{T}r(\rho_h) \to \mathbb{T}r(\rho_{ghg^{-1}}).$$

This map $\psi_g(h)$ is additive and multiplicative with respect to the direct sum and tensor product of 2-representations since it is given by composition of structural 2-morphisms of the 2-representation.

When ρ is a 2-representation in $2Vect_k$ and g and h commute, $\mathbb{T}r(\rho_h)$ and $\mathbb{T}r(\rho_{ghg^{-1}})$ are the same vector space. Thus Ganter and Kapranov introduce the following definition:

Definition 9. The *2-character* of ρ is a function on pairs of commuting elements:

$$\chi_\rho(h,g) = trace\big(\psi_g(h) : \mathbb{T}r(\rho_h) \to \mathbb{T}r(\rho_h)\big).$$

It is invariant under simultaneous conjugation.

Since $\mathbb{T}r$, $\psi_g(h)$, and $trace$ are all additive and multiplicative, we deduce that the character is also additive and multiplicative. (We will later give a different, explicit proof of this fact).

We are now prepared to compute the character of a 2-representation on $2Vect_{\mathbb{C}}$.

Theorem 10. *The 2-character of a 2-representation given by $\rho : G \to \Sigma_n$ and $c \in C^2(G; (\mathbb{C}^\times)^n_\rho)$ is*

$$\chi(h,g) = \sum_{i=\rho_g(i)=\rho_h(i)} c_i(g,g^{-1})^{-1} c_i(1,1)^{-1} c_i(h,g^{-1}) c_i(g,hg^{-1}).$$

Remark 11. This explicit formula can also be deduced from [2, Lemma 9.9] and [1, Lemma 30], by interpreting the computation of the map $\psi_g(h)$ in terms of the 2-cocycles of Proposition 4.

Proof. Let $\varphi : \mathbf{1}_{[n]} \Rightarrow \rho_h$, that is, a vector in $\mathbb{T}r(\rho_h)$. Note that this is an $n \times n$ matrix with zero entries everywhere except in those diagonal entries that are nonzero in ρ_h. Without loss of generality, we can assume those are in the first k rows (we can conjugate ρ by a permutation matrix f and change c accordingly to $f \cdot c$; this will just give a reordering of the indices which does not change the dimension of $\mathbb{T}r(\rho_h)$ nor the map $\psi_g(h)$). Thus we have

$$\varphi = \begin{bmatrix} a_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_k & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where k is the number of indices fixed by ρ_h.

We would like to compute now

$$\psi_g(h)(\varphi) = \phi_{g,h,g^{-1}} \cdot (\rho_g \circ \varphi \circ \rho_{g^{-1}}) \cdot \phi^{-1}_{g,g^{-1}} \cdot \phi^{-1}_1$$

which is a 2-morphism $\mathbf{1}_{[n]} \Rightarrow \rho_{ghg^{-1}}$. Note that we are omitting the associativity and unity isomorphisms.

For commuting pairs of elements (h,g), ρ_h and $\rho_g \circ \rho_h \circ \rho_{g^{-1}}$ are isomorphic, in particular, the nonzero entries in the diagonal are in the same position. Thus we have

$$\rho_g \circ \varphi \circ \rho_{g^{-1}} = \begin{bmatrix} a_{\rho_{g^{-1}}(1)} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{\rho_{g^{-1}}(k)} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

On the other hand, composition with the isomorphisms ϕ is given just by multiplication by the appropriate scalar in the appropriate row:

$$\psi_g(h) = \phi_{g,h,g^{-1}} \cdot (\rho_g \circ \varphi \circ \rho_{g^{-1}}) \cdot \phi^{-1}_{g,g^{-1}} \cdot \phi^{-1}_1 = \begin{bmatrix} b_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & b_k & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where

$$b_i = c_i(g, g^{-1})^{-1} c_i(1,1)^{-1} c_i(h, g^{-1}) c_i(g, hg^{-1}) a_{\rho_g^{-1}(i)}.$$

We can think of the matrices $\{e_i\}_{i=1}^k$, where e_i is the $n \times n$ matrix with 1 in the (i,i) entry and zero everywhere else, as a basis for $\mathbb{T}r(\rho_h)$. Then the contribution to the character comes from the indices i fixed by both ρ_h and ρ_g:

$$\chi(h,g) = \sum_{i=\rho_g(i)=\rho_h(i)} c_i(g,g^{-1})^{-1} c_i(1,1)^{-1} c_i(h,g^{-1}) c_i(g,hg^{-1}).$$

□

Remark 12. For a 1-dimensional representation we obtain that the character is given by

$$\chi(h,g) = c(g,g^{-1})^{-1} c(1,1)^{-1} c(h,g^{-1}) c(g,hg^{-1}),$$

where c is the 2-cocycle. This differs slightly from [5, Prop. 5.1] which gives

$$\chi(h,g) = c(g,g^{-1})^{-1} c(1,1)^{-1} c(g,h) c(gh,g^{-1}).$$

These seemingly different results are actually equal since c is a 2-cocycle, hence

$$c(h,g^{-1}) c(g,hg^{-1}) = c(g,h) c(gh,g^{-1}).$$

Corollary 13. *The 2-character of a 2-representation given by* $\rho : G \to \Sigma_n$ *and* $c \in C^2(G; (\mathbb{C}^\times)^n_\rho)$ *is*

$$\chi(h,g) = \sum_{i=\rho_g(i)=\rho_h(i)} \frac{c_i(h,g^{-1})}{c_i(g^{-1},h)}.$$

Proof. Since g and h commute and $[c]$ is a cocycle that

$$c_i(g, hg^{-1}) = c_i(g, g^{-1}h) = \frac{c_i(1,h)c_i(g,g^{-1})}{c_i(g^{-1},h)}.$$

The cocycle condition also implies that $c_i(1,h) = c_i(1,1)$.
Substituting in the formula of Theorem 10, we obtain the result. $\square$

Lemma 14. *The character is invariant under equivalence.*

Proof. Given two equivalent representations given by ρ, $[c] \in H^2(G; (\mathbb{C}^\times)^n_\rho)$ and ρ', $[c'] \in H^2(G; (\mathbb{C}^\times)^n_{\rho'})$, there exists $f \in \Sigma_n$ such that $\rho' = f^{-1}\rho f$ and $[c'] = [f \cdot c]$, with $(\delta b)(g,h) = g \cdot b(h) - b(gh) + b(g) = c'(g,h) - f \cdot c(g,h)$.
Then

$$\begin{aligned}
\chi'(h,g) &= \sum_{i=\rho'_g(i)=\rho'_h(i)} c'_i(g,g^{-1})^{-1}c'_i(1,1)^{-1}c'_i(h,g^{-1})c'_i(g,hg^{-1}) \\
&= \sum_{i=f\rho_g f^{-1}(i)=f\rho_h f^{-1}(i)} \frac{b_i(1)}{c_{f^{-1}(i)}(g,g^{-1})b_i(g^{-1})b_i(g)} \cdot \frac{b_i(1)}{c_{f^{-1}(i)}(1,1)b_i(1)b_i(1)} \\
&\qquad \cdot \frac{c_{f^{-1}(i)}(h,g^{-1})b_i(g^{-1})b_i(h)}{b_i(hg^{-1})} \cdot \frac{c_{f^{-1}(i)}(g,hg^{-1})b_i(hg^{-1})b_i(g)}{b_i(h)} \\
&= \sum_{i=\rho_g(i)=\rho_h(i)} c_i(g,g^{-1})^{-1}c_i(1,1)^{-1}c_i(h,g^{-1})c_i(g,hg^{-1}) \\
&= \chi(h,g).
\end{aligned}$$

$\square$

Lemma 15. *The character respects the additive and multiplicative structure of representations.*

Proof. Let $\rho, [c]$ and $\rho', [c']$ represent two representations of dimensions n, n'.. The direct sum representation is given by $\tilde{\rho}_h$, the block sum of the matrices ρ_h and ρ'_h for every h and the cocycle $\tilde{c}$ with

$$\tilde{c}_i = \begin{cases} c_i & \text{if } i \leq n, \\ c'_{i-n} & \text{if } i > n. \end{cases}$$

The character of the direct sum is

$$
\begin{aligned}
\tilde{\chi}(h,g) &= \sum_{i=\tilde{\rho}_g(i)=\tilde{\rho}_h(i)} \tilde{c}_i(g,g^{-1})^{-1}\tilde{c}_i(1,1)^{-1}\tilde{c}_i(h,g^{-1})\tilde{c}_i(g,hg^{-1}) \\
&= \sum_{n\geq i=\rho_g(i)=\rho_h(i)} c_i(g,g^{-1})^{-1}c_i(1,1)^{-1}c_i(h,g^{-1})c_i(g,gh^{-1}) \\
&+ \sum_{\substack{i>n \\ i-n=\rho'_g(i-n)=\rho'_h(i-n)}} c'_{i-n}(g,g^{-1})^{-1}c'_{i-n}(1,1)^{-1}c'_{i-n}(h,g^{-1})c'_{i-n}(g,hg^{-1}) \\
&= \chi(h,g)+\chi'(h,g).
\end{aligned}
$$

On the other hand, let $\overline{\rho}$, $[\overline{c}]$ denote the tensor product of ρ, $[c]$ and ρ', $[c']$. From the definition of the tensor product and using the labeling above for the set of nn' elements, it is not hard to see that

$$
(\overline{\rho}_g)_{(i,i'),(j,j')} = (\rho_g)_{i,j}(\rho'_g)_{i',j'},
$$

$$
\overline{c}_{(i.i')} = c_i c'_{i'}.
$$

Then (i,i') is fixed by $\overline{\rho}_h$ if and only if i is fixed by ρ_h and i' is fixed by ρ'_h.
Thus

$$
\begin{aligned}
\overline{\chi}(h,g) &= \sum_{(i,i')=\overline{\rho}_g(i,i')=\overline{\rho}_h(i,i')} \overline{c}_{(i,i')}(g,g^{-1})^{-1}\overline{c}_{(i,i')}(1,1)^{-1}\overline{c}_{(i,i')}(h,g^{-1})\overline{c}_{(i,i')}(g,hg^{-1}) \\
&= \sum_{\substack{(i,i') \\ i=\rho_g(i)=\rho_h(i) \\ i'=\rho_g(i')=\rho_h(i')}} ((c_ic'_{i'})(g,g^{-1})(c_ic'_{i'})(1,1))^{-1}(c_ic'_{i'})(h,g^{-1})(c_ic'_{i'})(g,hg^{-1}) \\
&= \left(\sum_{i=\rho_g(i)=\rho_h(i)} c_i(g,g^{-1})^{-1}c_i(1,1)^{-1}c_i(h,g^{-1})c_i(g,hg^{-1})\right) \\
&\quad\cdot\left(\sum_{i'=\rho'_g(i')=\rho'_h(i')} c'_{i'}(g,g^{-1})^{-1}c'_{i'}(1,1)^{-1}c'_{i'}(h,g^{-1})c'_{i'}(g,hg^{-1})\right) \\
&= \chi(h,g)\chi'(h,g).
\end{aligned}
$$

Hence we see that the characters respect the additive an multiplicative structures on the representations.

$\square$

Remark 16. One can also use Theorem 10 to reproduce the formula for the character of the induced representation which appears in [5, Corollary 7.6]:

$$
\chi_{\text{ind}}(h,g) = \frac{1}{|H|}\sum_{s^{-1}(h,g)s\in H\times H} \chi(s^{-1}hs, s^{-1}gs).
$$

On the other hand, using Proposition 7, [5, Prop. 5.1, Cor. 7.6] and the additivity of the 2-character, we can recover the result of Theorem 10.

One might hope that, analogous to the case of 1-characters, the map from equivalence classes of 2-representations to characters is injective. This turns out not to be true, see [2, Corollary 9.11]. Here is an explicit counterexample.

Example. We will consider two 2-representations of Σ_3 of dimension 8 with trivial cocycle, so they amount to a group homomorphism $\rho : \Sigma_3 \to \Sigma_8$, that is, they are permutation representations. Note also that they are isomorphic are permutations representations if and only if they are isomorphic as 2-representations. Since they have trivial cocycle, the character is given by

$$\chi(h, g) = \sum_{i=\rho_g(i)=\rho_h(i)} 1 = \#\{i = \rho_g(i) = \rho_h(i)\}.$$

Let ρ be given by three blocks: the regular representation (action of Σ_3 on itself), and two trivial blocks. Let ρ' be given by three blocks as well: 2 blocks with the action of Σ_3 on $\Sigma_3/\langle(12)\rangle$ and one block with the action of Σ_3 on $\Sigma_3/\langle(123)\rangle$.

Note that these two 2-representations are not isomorphic: Σ_3 fixes the last two elements of ρ while Σ_3 fixes no element of ρ'.

On the other hand, we can prove that these two representations have the same character: the pairs of commuting elements in Σ_3 are those containing 1, those with $g = h$, and $\{(123), (132)\}$.

We can directly compute the characters:

$$\begin{aligned} \chi(1,1) = \chi'(1,1) &= 8; \\ \chi(1,g) = \chi'(1,g) &= 2 \quad \text{for all} \quad g \neq 1; \\ \chi(g,g) = \chi'(g,g) &= 2 \quad \text{for all} \quad g \neq 1; \\ \chi((123),(132)) = \chi'((123),(132)) &= 2. \end{aligned}$$

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