Uniform Boundedness

01° Let \( z = x + iy \) be a complex number for which the real part \( x \) is positive. Let us form the sequence:

\[
(\bullet) \quad w_0 = z, \quad w_1 = z + \frac{1}{w_0}, \quad w_2 = z + \frac{1}{w_1}, \quad \ldots, \quad w_{k+1} = z + \frac{1}{w_k}, \quad \ldots
\]

By induction, it is plain that, for each nonnegative integer \( k \), the real parts of \( w_k \) and \( 1/w_k \) are positive. Hence:

\[
(1) \quad x \leq |w_k|
\]

It follows that:

\[
|w_{k+1}| = |z + \frac{1}{w_k}| \leq |z| + \frac{1}{|w_k|} \leq |z| + \frac{1}{x}
\]

Hence:

\[
(2) \quad x \leq |w_k| \leq |z| + \frac{1}{x}
\]

02° Now let \( K \) be a compact set of complex numbers such that, for each \( z \) in \( K \), the real part \( x \) of \( z \) is positive. Let \( b \) be a positive number such that, for each \( z \) in \( K \):

\[
|z| + \frac{1}{x} \leq b
\]

By the foregoing observations, it is plain that, for each \( z \) in \( K \), the sequence \((\bullet)\) defined by \( z \) is bounded by \( b \).

03° Obviously, if the sequence \((\bullet)\) is convergent then the limit \( w \) must satisfy the relation:

\[
(\circ) \quad w^2 - zw - 1 = 0
\]

Of course, \( w \) must then be the zero for which the real part is positive. By the “subsubsequence” argument, the sequence \((\bullet)\) must in fact converge to \( w \).

04° In context of Complex Analysis, we infer that sequence of analytic functions of \( z \), defined by \((\bullet)\) in the right half plane, converges uniformly on compact sets to the limit defined by \((\circ)\).