COMPLEX DIFFERENTIATION

01° We identify \( \mathbb{C} \) with \( \mathbb{R}^2 \), subject to the following notation:

\[
z = x + iy = \begin{pmatrix} x \\ y \end{pmatrix}
\]

Let \( W \) be a region in \( \mathbb{C} \) and let \( f \) be a mapping carrying \( W \) to \( \mathbb{C} \):

\[
f(z) = w = u + iv = \begin{pmatrix} u \\ v \end{pmatrix} \quad (z \in W)
\]

By the foregoing identification, we may regard \( f \) as a mapping carrying \( W \) to \( \mathbb{R}^2 \):

\[
f\left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} u \\ v \end{pmatrix}
\]

02° Let \( z_o \) be a member of \( W \):

\[
z_o = x_o + iy_o = \begin{pmatrix} x_o \\ y_o \end{pmatrix}
\]

One says that \( f \) is analytic at \( z_o \) iff there is a member \( c \) of \( \mathbb{C} \):

\[
c = a + ib = \begin{pmatrix} a \\ b \end{pmatrix}
\]

such that:

\[
(1) \quad \lim_{z \to z_o} \frac{1}{z - z_o} (f(z) - f(z_o) - c(z - z_o)) = 0
\]

One says that \( f \) is totally differentiable at \( z_o \) iff there is a two by two matrix \( M \):

\[
M = \begin{pmatrix} p & r \\ q & s \end{pmatrix}
\]

having real entries such that:

\[
(2) \quad \lim_{(x, y) \to (x_o, y_o)} \frac{1}{\| (x - x_o, y - y_o) \|} \| f\left( \begin{pmatrix} x \\ y \end{pmatrix} \right) - f\left( \begin{pmatrix} x_o \\ y_o \end{pmatrix} \right) - M \left( \begin{pmatrix} x - x_o \\ y - y_o \end{pmatrix} \right) \| = 0
\]
In the latter case, it might (but may not) happen that \( s = p \) and \( r = -q \):

\[
M = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}
\]

03° Given a member \( c \) of \( \mathbb{C} \):

\[
c = a + ib
\]

we may introduce the following two by two matrix \( M \):

\[
M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}
\]

having real entries. Clearly:

\[
\begin{aligned}
cz &= (ax - by) + i(bx + ay) = \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}
\]

04° Now it is plain that (1) holds iff (2) and (3) hold, where \( c \) and \( M \) are linked by (4) and (5). We conclude that \( f \) is analytic at \( z_o \) iff \( f \) is totally differentiable at \( z_o \) and the following relations hold:

\[
\begin{aligned}
(CR) & \quad \frac{\partial u}{\partial x}(x_o, y_o) = \frac{\partial v}{\partial y}(x_o, y_o), \quad \frac{\partial u}{\partial y}(x_o, y_o) = -\frac{\partial v}{\partial x}(x_o, y_o) \\
\end{aligned}
\]

One calls these relations the Cauchy/Riemann Equations. Obviously:

\[
\begin{aligned}
\frac{d}{dx}(z_o) = \frac{\partial u}{\partial x}(x_o, y_o) + i \frac{\partial v}{\partial x}(x_o, y_o) = \frac{\partial v}{\partial y}(x_o, y_o) - i \frac{\partial u}{\partial y}(x_o, y_o)
\end{aligned}
\]

05° Informally, we write the foregoing relations as follows:

\[
(CR) \quad u_x = v_y, \quad u_y = -v_x
\]

and:

\[
(7) \quad f' = u_x + iv_x = v_y - iu_y
\]