01° Let $\Delta$ be the open disk in $\mathbb{C}$ centered at 0 with radius 1. Let $D$ be the closure of $\Delta$. Let $f$ be a complex-valued function defined on $D$, continuous on $D$, and analytic on $\Delta$. Assume that there is a positive real number $\epsilon$ such that:

$$0 \leq \theta \leq \epsilon \implies f(e^{i\theta}) = 0$$

Show that $f$ must be constantly 0 on $D$.

02° Let $n$ be an integer for which $2 < n$ and let $f$ be the polynomial defined as follows:

$$f(z) = z^n - \frac{1}{4}(1 + z + z^2) \quad (z \in \mathbb{C})$$

Show that the zeros of $f$ lie in the unit disk $\Delta$.

03° For any complex numbers $\zeta_1$ and $\zeta_2$, let us write:

$$\zeta_1 \cdot \zeta_2 = \Re(\zeta_1 \bar{\zeta}_2) = \Re(\zeta_1)\Re(\zeta_2) + \Im(\zeta_1)\Im(\zeta_2)$$

For each $z$ in the unit disk $\Delta$ and for any complex numbers $\zeta_1$ and $\zeta_2$, let us write:

$$\langle\langle \zeta_1, \zeta_2 \rangle\rangle_z = \left(\frac{2}{1 - |z|^2}\right)^2 \zeta_1 \cdot \zeta_2$$

Now let $H$ be an automorphism of $\Delta$:

$$H(z) = \frac{\alpha z + \beta}{\beta z + \alpha} \quad (z \in \Delta)$$

where $|\alpha|^2 - |\beta|^2 = 1$. Let $z$ be any member of $\Delta$ and let $\zeta_1$ and $\zeta_2$ be any complex numbers. Let:

$$w = H(z), \ \eta_1 = H'(z)\zeta_1, \ \eta_2 = H'(z)\zeta_2$$

Show that:

$$\langle\langle \eta_1, \eta_2 \rangle\rangle_w = \langle\langle \zeta_1, \zeta_2 \rangle\rangle_z$$

[If you like, you may restrict your attention to the case in which:

$$\alpha = \cosh(\theta), \ \beta = \sinh(\theta)$$

where $\theta$ is any real number.] Conclude that $H$ is an isometry for the metric structure $\langle\langle \cdot, \cdot \rangle\rangle$ on $\Delta$. 