01 • The Bernoulli Numbers are defined by the following relation:

\[ \frac{z}{\exp(z) - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k \]

Find \( B_0, B_1, B_2, B_3, \) and \( B_4. \) Show that for any integer \( k, \) if \( 3 \leq k \) and if \( k \) is odd then \( B_k = 0. \)

02 • The Fibonacci Numbers are defined by the following relation:

\[ \frac{z}{1 - z - z^2} = \sum_{j=0}^{\infty} F_j z^j \]

Find the real numbers \( a \) and \( b \) for which:

\[ 1 - z - z^2 = (1 - az)(1 - bz), \quad a < 0 < b \]

Show that:

\[ F_j = \frac{b^j - a^j}{\sqrt{5}} \quad (j = 0, 1, 2, 3, \ldots) \]

To that end, first find the real numbers \( A \) and \( B \) such that:

\[ \frac{z}{1 - z - z^2} = \frac{A}{1 - az} + \frac{B}{1 - bz} \]

Find the radius of convergence for the power series.

03 • Let \( z \) be any complex number and let \( r \) be any positive real number. Let \( C_r(z) \) stand for the circle having center \( z \) and radius \( r: \)

\[ w \in C_r(z) \iff |w - z| = r \]

and let \( D_r(z) \) stand for the open disk having center \( z \) and radius \( r: \)

\[ w \in D_r(z) \iff |w - z| < r \]

Let \( \gamma_r(z) \) be the curve, parametrized by arclength, which traverses \( C_r(z) \) once ccw:

\[ \gamma_r(z)(s) = z + r \exp(i - s), \quad (0 \leq s \leq 2\pi r) \]
Let $\Omega$ be a region in $\mathbb{C}$ which includes:

$$\text{clo}(D_r(z)) = D_r(z) \cup C_r(z)$$

and let $f$ be a function defined and analytic on $\Omega$. Apply the Cauchy Integral Formula, for circular curves, to show that:

$$f(z) = \frac{1}{2\pi r} \int_{0}^{2\pi r} f(z + re^{i\frac{s}{r}})ds$$

Hence, $f(z)$ is the average of the values of $f$ on $C_r(z)$. Apply polar coordinates to show that:

$$f(z) = \frac{1}{\pi r^2} \int \int_{D_z(r)} f(x + iy)dxdy$$

Hence, $f(z)$ is the average of the values of $f$ on $D_r(z)$.

04 $\bullet$ Let $r$ be a positive real number for which $0 < r < 1$. Let $f$ be the function defined as follows:

$$f(z) = \frac{1}{1 - rz} \quad (|z| < \frac{1}{r})$$

Obviously, $f$ is analytic. Let $C$ be the circle having center 0 and radius 1. Let $\gamma$ be the curve which traverses $C$ once ccw:

$$\gamma(\theta) = exp(i\theta), \quad (0 \leq \theta \leq 2\pi)$$

Apply the Cauchy Integral Formula, for circular curves, to show that:

$$\frac{1}{1 - r^2} = f_r(r) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1}{1 - 2rcos(\theta) + r^2} d\theta$$