THE WAVE EQUATION IN THREE DIMENSIONS
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The Homogeneous Wave Equation

01° Let $f$ and $g$ be complex valued functions defined on $\mathbb{R}^3$. We propose to solve the Homogeneous Wave Equation:

$$(\diamond) \quad \dot{\gamma}(t, x, y, z) - (\Delta \gamma)(t, x, y, z) = 0$$

subject to the Initial Conditions:

$$\gamma(0, x, y, z) = f(x, y, z), \quad \gamma_t(0, x, y, z) = g(x, y, z)$$

Of course, $\gamma$ is the complex valued function defined on $\mathbb{R}^4$, required to be found. To be clear, we recall that:

$$(\Delta \gamma)(t, x, y, z) \equiv \gamma_{xx}(t, x, y, z) + \gamma_{yy}(t, x, y, z) + \gamma_{zz}(t, x, y, z)$$

The Method of Fourier: Spherical Means

02° We pass to the Fourier Transform of $\gamma$:

$$(\phi) \quad \hat{\gamma}(t, u, v, w) = \int \int \int_{\mathbb{R}^3} \gamma(t, x, y, z)e^{-i(ux + vy + wz)}m(dxdydz)$$

$$\gamma(t, x, y, z) = \int \int \int_{\mathbb{R}^3} \hat{\gamma}(t, u, v, w)e^{i(ux + vy + wz)}m(dudvdw)$$

In the foregoing relations, we have adopted the following notational convention:

$$m(dudvdw) = \frac{1}{(2\pi)^{3/2}}dudvdw, \quad m(dxdydz) = \frac{1}{(2\pi)^{3/2}}dxdydz$$

Clearly:

$$\gamma_t(t, x, y, z) = \int \int \int_{\mathbb{R}^3} \hat{\gamma}_{tt}(t, u, v, w)e^{i(ux + vy + wz)}m(dudvdw)$$

$$(\Delta \gamma)(t, x, y, z) = \int \int \int_{\mathbb{R}^3} -u^2 - v^2 - w^2 \hat{\gamma}(t, u, v, w)e^{i(ux + vy + wz)}m(dudvdw)$$
We obtain the following reformulation of equations (a) and (b):

(a) \[ \dot{\gamma}(t, u, v, w) + (u^2 + v^2 + w^2)\dot{\gamma}(t, u, v, w) = 0 \]

(b) \[ \dot{\gamma}(0, u, v, w) = f(u, v, w), \quad \dot{\gamma}_u(0, u, v, w) = g(u, v, w) \]

Now \( \dot{\gamma} \) must take the form:

\[ \dot{\gamma}(t, u, v, w) = f(u, v, w) \cos(\sqrt{u^2 + v^2 + w^2} t) \]

\[ + g(u, v, w) \frac{1}{\sqrt{u^2 + v^2 + w^2}} \sin(\sqrt{u^2 + v^2 + w^2} t) \]

03° We need to recover \( \gamma \) from \( \dot{\gamma} \). To that end, let \( h \) be a complex valued function defined on \( \mathbb{R}^3 \). Let \( \nu_h \) be the complex valued function defined on \( \mathbb{R}^4 \) as follows:

(1) \[ \nu_h(t, u, v, w) \equiv \frac{1}{\sqrt{u^2 + v^2 + w^2}} \sin(\sqrt{u^2 + v^2 + w^2} t) \]

Obviously:

\[ \dot{\gamma}(t, u, v, w) = \partial_t t \nu_f(t, u, v, w) + t \nu_g(t, u, v, w) \]

Let \( \mu_h \) be the complex valued function defined on \( \mathbb{R}^4 \) as follows:

(2) \[ \mu_h(t, x, y, z) \equiv \int \int \int_{\mathbb{R}^3} \nu_h(t, u, v, w) e^{ixu + iyv + izw} m(dudvdw) \]

Obviously:

\[ \gamma(t, x, y, z) = \partial_t t \mu_f(t, x, y, z) + t \mu_g(t, x, y, z) \]

04° To check that the foregoing solution of the Wave Equation meets the required initial conditions, we note that, by definition (1):

\[ tv_h(t, u, v, w)|_{t=0} = 0 \]

\[ \frac{\partial}{\partial t} tv_h(t, u, v, w)|_{t=0} = \dot{h}(u, v, w) \]

\[ \frac{\partial^2}{\partial t^2} tv_h(t, u, v, w)|_{t=0} = 0 \]
But we need to present $\mu_f$ and $\mu_g$ in a more perspicuous form. To that end, we contend that:

$$\frac{1}{\sqrt{u^2 + v^2 + w^2}} \sin(\sqrt{u^2 + v^2 + w^2}) = \frac{1}{4\pi} \int_{\Sigma} e^{i(u\bar{x} + v\bar{y} + w\bar{z})} \cos(\theta) d\phi d\theta$$

where $\Sigma$ is the unit sphere in $\mathbb{R}^3$ and where:

$$\bar{x} = \cos(\theta)\cos(\phi)$$
$$\bar{y} = \cos(\theta)\sin(\phi)$$
$$\bar{z} = \sin(\theta)$$

For now, let us assume that relation (3) holds. [See article 8°.]

Clearly, for any positive number $t$, we have:

$$\frac{1}{\sqrt{u^2 + v^2 + w^2}} \sin(\sqrt{u^2 + v^2 + w^2}t) = \frac{1}{4\pi t^2} \int_{\Sigma} e^{i(u\bar{x} + v\bar{y} + w\bar{z})t^2} \cos(\theta) d\phi d\theta$$

Consequently:

$$\nu_h(t, u, v, w) = \hat{h}(u, v, w) \frac{1}{4\pi t^2} \int_{\Sigma} e^{i(u\bar{x} + v\bar{y} + w\bar{z})t^2} \cos(\theta) d\phi d\theta$$

so that:

$$\mu_h(t, x, y, z) = \int_{\mathbb{R}^3} \nu_h(t, u, v, w)e^{i(ux + vy + wz)}m(du dv dw)$$
$$= \frac{1}{4\pi t^2} \int_{\Sigma} \hat{h}(x + t\bar{x}, y + t\bar{y}, z + t\bar{z})t^2 \cos(\theta) d\phi d\theta$$

Clearly, $\mu_h(t, x, y, z)$ is the average value of $h$ over the sphere of radius $t$ centered at $(x, y, z)$.

Obviously, the foregoing relation is sensible for any value of $t$. One refers to $\mu_h$ as the Spherical Mean defined by $h$. Now we can present the solution $\gamma$ of the Wave Equation in terms of Spherical Means, as follows:

$$\gamma(t, x, y, z) = \frac{\partial}{\partial t} \frac{t}{4\pi t^2} \int_{\Sigma} f(x + t\bar{x}, y + t\bar{y}, z + t\bar{z})t^2 \cos(\theta) d\phi d\theta$$
$$+ \frac{1}{4\pi t^2} \int_{\Sigma} g(x + t\bar{x}, y + t\bar{y}, z + t\bar{z})t^2 \cos(\theta) d\phi d\theta$$
Finally, let us prove relation (3). For that purpose, let us introduce the function
\[ \phi(u, v, w) = \frac{1}{4\pi} \int_\Sigma e^{+i(u\bar{x}+v\bar{y}+w\bar{z})}\cos(\theta)d\phi d\theta \]
which represents the right hand side of the relation. Obviously, \( \phi \) is invariant under rotations, so we may present \( \phi \) as follows:
\[ \phi(u, v, w) = \psi(s) \quad (0 < s = \sqrt{u^2+v^2+w^2}) \]
Moreover:
\[ (\triangle \phi)(u, v, w) = -\frac{1}{4\pi} \int_\Sigma (\bar{x}^2 + \bar{y}^2 + \bar{z}^2)e^{+i(u\bar{x}+v\bar{y}+w\bar{z})}\cos(\theta)d\phi d\theta 
= -\phi(u, v, w) \]
so that:
\[ \psi^{\circ\circ}(s) + 2s\psi^\circ(s) = -\psi(s) \]
Under the transformation \( \chi(s) = s\psi(s) \), we find that:
\[ \chi^{\circ\circ}(s) = -\chi(s) \]
Consequently, there must be complex numbers \( \alpha \) and \( \beta \) such that:
\[ \psi(s) = \alpha \frac{1}{s}\cos(s) + \beta \frac{1}{s}\sin(s) \]
However:
\[ \lim_{s \to 0} \psi(s) = 1 \]
Therefore, \( \alpha = 0, \beta = 1 \), and:
\[ \psi(s) = \frac{1}{s}\sin(s) \]
The proof of relation (3) is complete.

Energy

Let \( \gamma \) be a solution of the Homogeneous Wave Equation:
\[ \gamma_{tt}(t, x, y, z) - (\triangle \gamma)(t, x, y, z) = 0 \]
such that the Initial Conditions:
\[ \gamma(0, x, y, z) = f(x, y, z), \quad \gamma_t(0, x, y, z) = g(x, y, z) \]
Let $\epsilon$ be the function defined on $\mathbb{R}^4$ as follows:

$$
\epsilon(t, x, y, z) \\
\equiv \frac{1}{2} (|\gamma_t(t, x, y, z)|^2 + |\gamma_x(t, x, y, z)|^2 + |\gamma_y(t, x, y, z)|^2 + |\gamma_z(t, x, y, z)|^2)
$$

One refers to $\epsilon$ as the Energy Density. We contend that the corresponding Energy Integral:

$$
\eta(t) = \iiint_{\mathbb{R}^3} \epsilon(t, x, y, z) m (dxdydz)
$$

is constant. To prove the contention, we call upon several cases of Parseval’s Relation:

$$
\iiint_{\mathbb{R}^3} |\gamma_t(t, x, y, z)|^2 m (dxdydz) = \iiint_{\mathbb{R}^3} |\hat{\gamma}_t(t, u, v, w)|^2 m (dudvdw)
$$

$$
\iiint_{\mathbb{R}^3} (|\gamma_x(t, x, y, z)|^2 + |\gamma_y(t, x, y, z)|^2 + |\gamma_z(t, x, y, z)|^2) m (dxdydz)
$$

$$
= \iiint_{\mathbb{R}^3} (u^2 + v^2 + w^2) |\hat{\gamma}(t, u, v, w)|^2 m (dudvdw)
$$

From article 2°, we recover the relations:

$$
\hat{\gamma}(t, u, v, w)
$$

$$
= \hat{f}(u, v, w) \cos(\sqrt{u^2 + v^2 + w^2} t)
$$

$$
+ \hat{g}(u, v, w) \frac{1}{\sqrt{u^2 + v^2 + w^2}} \sin(\sqrt{u^2 + v^2 + w^2} t)
$$

$$
\hat{\gamma}_t(t, u, v, w)
$$

$$
= -\hat{f}(u, v, w) \sqrt{u^2 + v^2 + w^2} \sin(\sqrt{u^2 + v^2 + w^2} t)
$$

$$
+ \hat{g}(u, v, w) \cos(\sqrt{u^2 + v^2 + w^2} t)
$$

Let us write $s$ for $\sqrt{u^2 + v^2 + w^2}$, $C$ for $\cos(st)$, and $S$ for $\sin(st)$. Also, let us drop display of the variables $u$, $v$, and $w$. Now we have:

$$
|\hat{\gamma}|^2 = (\hat{f}C + \hat{g}s)(\hat{f}C + \hat{g}s)
$$

$$
|\hat{\gamma}_t|^2 = (-\hat{f}sS + \hat{g}C)(-\hat{f}sS + \hat{g}C)
$$

By straightforward computation, we find that:

$$
|\hat{\gamma}_t|^2 + s^2 |\hat{\gamma}|^2 = |\hat{g}|^2 + s^2 |\hat{f}|^2
$$
Hence:

\[
2\eta(t) = \iiint_{\mathbb{R}^3} (|\dot{\gamma}(t, u, v, w)|^2 + (u^2 + v^2 + w^2)|\ddot{\gamma}(t, u, v, w)|^2)\, m(du dv dw)
\]

\[
= \iiint_{\mathbb{R}^3} (|\dot{g}(u, v, w)|^2 + (u^2 + v^2 + w^2)|\ddot{f}(u, v, w)|^2)\, m(du dv dw)
\]

Obviously, \(\eta\) is constant. In fact:

\[
(\epsilon) \quad \eta(t) = \frac{1}{2} \iiint_{\mathbb{R}^3} (|g(x, y, z)|^2 + |(\nabla f)(x, y, z)|^2)\, m(dx dy dz)
\]

### A Particular Solution of the Inhomogeneous Wave Equation

10° Let \(\delta\) be a complex valued function defined on \(\mathbb{R}^4\). We propose to solve the Inhomogeneous Wave Equation:

\[(\circ) \quad \gamma_{tt}(t, x, y, z) - (\triangle \gamma)(t, x, y, z) = \delta(t, x, y, z)\]

subject to the particular Initial Conditions:

\[(\bullet) \quad \gamma(0, x, y, z) = 0, \quad \gamma_t(0, x, y, z) = 0\]

To that end, we introduce the complex valued function \(\beta\) defined on \(\mathbb{R}^5\) as follows:

\[
\beta(s, t, x, y, z) \equiv \frac{t}{4\pi t^2} \iiint_\Sigma \delta(s, x + t\bar{x}, y + t\bar{y}, z + t\bar{z})t^2\cos(\theta)d\phi d\theta
\]

With reference to our prior development of Spherical Means, we find that, for each \(s\):

\[
(5) \quad \beta_{tt}(s, t, x, y, z) - (\triangle \beta)(s, t, x, y, z) = 0
\]

\[
(6) \quad \beta(s, 0, x, y, z) = 0, \quad \beta_t(s, 0, x, y, z) = \delta(s, x, y, z)
\]

In turn, let \(\gamma\) be the complex valued function defined on \(\mathbb{R}^4\) as follows:

\[
(\ast) \quad \gamma(t, x, y, z) \equiv \int_0^t \beta(s, t - s, x, y, z)\, ds
\]

Let us verify that \(\gamma\) satisfies the foregoing conditions (\(\circ\)) and (\(\bullet\)).
11° We note first that:

\[ \gamma(0, x, y, z) = \int_0^0 \beta(s, -s, x, y, z) ds = 0 \]

By differentiation with respect to \( t \), we find that:

\[
\gamma_t(t, x, y, z) = \beta(t, 0, x, y, z) + \int_0^t \beta_t(s, t - s, x, y, z) ds \\
= 0 + \int_0^t \beta_t(s, t - s, x, y, z) ds
\]

Obviously:

\[ \gamma_t(0, x, y, z) = \int_0^0 \beta_t(s, -s, x, y, z) ds = 0 \]

Again, by differentiation with respect to \( t \), we find that:

\[
\gamma_{tt}(t, x, y, z) = \beta_t(t, 0, x, y, z) + \int_0^t \beta_{tt}(s, t - s, x, y, z) ds
\]

Finally, by appropriate differentiations with respect to \( x \), \( y \), and \( z \), we find that:

\[(\Delta \gamma)(t, x, y, z) = \int_0^t (\Delta \beta)(s, t - s, x, y, z) ds\]

Now relations (5) and (6) yield conditions (\( \circ \)) and (\( \bullet \)).

**The General Solution of the Inhomogeneous Wave Equation**

12° Let \( \delta \) be a complex valued function defined on \( \mathbb{R}^4 \) and let \( f \) and \( g \) be complex valued functions defined on \( \mathbb{R}^3 \). Let us solve the Inhomogeneous Wave Equation:

\[
(\circ) \quad \gamma_{tt}(t, x, y, z) - (\Delta \gamma)(t, x, y, z) = \delta(t, x, y, z)
\]

subject to the Initial Conditions:

\[
(\bullet) \quad \gamma(0, x, y, z) = f(x, y, z), \quad \gamma_t(0, x, y, z) = g(x, y, z)
\]

Actually, we need to say very little. One may obtain a solution \( \gamma \) by adding the solutions to the foregoing cases, displayed in articles 7° and 10°.
Uniqueness

13° In context of the foregoing article, let us consider two solutions \( \gamma_1 \) and \( \gamma_2 \) of the Inhomogeneous Wave Equation (\( \circ \)), both of which meet the Initial Conditions (\( \bullet \)). Let \( \gamma \equiv \gamma_1 - \gamma_2 \). Obviously, \( \gamma \) is a solution of the Homogeneous Wave Equation:

\[
\gamma_{tt}(t, x, y, z) - (\Delta \gamma)(t, x, y, z) = 0
\]

and it satisfies the Initial Conditions:

\[
\gamma(0, x, y, z) = 0, \quad \gamma_t(0, x, y, z) = 0
\]

By article 2°, it is plain that \( \hat{\gamma} = 0 \). Hence, \( \gamma = 0 \). Therefore, \( \gamma_1 = \gamma_2 \).

Rigour

14° In the foregoing articles, we have applied the Fourier Transform and the operations of differentiation and integration in a manner somewhat cavalier. We need to be more precise.

15° Let \( S \) be the complex linear space consisting of all smooth complex valued functions:

\[
h(x, y, z)
\]

defined on \( \mathbb{R}^3 \) which are are rapidly decreasing in \( x \), \( y \), and \( z \). We mean to say that, for any nonnegative integers \( p \), \( a \), \( b \), and \( c \), the function:

\[
(1 + x^2 + y^2 + z^2)^p \frac{\partial^{a+b+c}}{\partial x^a \partial y^b \partial z^c} h(x, y, z)
\]

defined on \( \mathbb{R}^3 \) is bounded. In turn, let \( W \) be the complex linear space consisting of all smooth complex valued functions:

\[
\gamma(t, x, y, z)
\]

defined on \( \mathbb{R}^4 \) which are are rapidly decreasing in \( x \), \( y \), and \( z \), locally uniformly in \( t \). We mean to say that, for any finite interval \( V \) in \( \mathbb{R} \) and for any nonnegative integers \( p \), \( \ell \), \( a \), \( b \), and \( c \), the restriction of the function:

\[
(1 + x^2 + y^2 + z^2)^p \frac{\partial^{\ell+a+b+c}}{\partial t^\ell \partial x^a \partial y^b \partial z^c} \gamma(t, x, y, z)
\]

defined on \( \mathbb{R}^4 \) to the set \( J \times \mathbb{R}^3 \) is bounded.

16° For functions in \( S \) or \( W \), the Fourier Transform and its inverse are well defined.
17° Obviously, for each function $\gamma$ in $W$, the function:

$$\gamma \equiv \gamma_{tt} - \Delta \gamma$$

also lies in $W$. Consequently, we may introduce the Wave Operator $\Box$, a linear mapping carrying $W$ to itself:

$$\Box \gamma \quad (\gamma \in W)$$

18° Now let $K$ be the linear subspace of $W$ defined by the following condition:

$$\gamma \in K \iff \Box \gamma = 0$$

Of course, $K$ is the kernel of $\Box$. With reference to articles 2° and 7°, we may presume to introduce a linear mapping $\Gamma$ carrying $S \times S$ to $K$:

$$\Gamma(f, g) \equiv \gamma \quad ((f, g) \in S \times S)$$

defined in terms of spherical means as follows:

$$\gamma(t, x, y, z)$$

$$\equiv \frac{\partial}{\partial t} t \int_{\Sigma} f(x + t \bar{x}, y + t \bar{y}, z + t \bar{z}) t^2 \cos(\theta) d\phi d\theta$$

$$+ \frac{t}{4\pi t^2} \int_{\Sigma} g(x + t \bar{x}, y + t \bar{y}, z + t \bar{z}) t^2 \cos(\theta) d\phi d\theta$$

To justify the definition of $\Gamma$, we must show that $\gamma$ lies in $W$. It will follow, by design, that $\gamma$ lies in $K$. To that end, let us observe that, for each function $h$ in $S$:

$$\frac{\partial^\ell}{\partial t^\ell} h(x + t \bar{x}, y + t \bar{y}, z + t \bar{z})$$

$$= \sum_{a+b+c=\ell} \frac{\ell!}{a!b!c!} \frac{\partial^{a+b+c}}{\partial x^a \partial y^b \partial z^c} h(x + t \bar{x}, y + t \bar{y}, z + t \bar{z}) \bar{x}^a \bar{y}^b \bar{z}^c$$

Let us also observe that:

$$(1 + x^2 + y^2 + z^2)$$

$$\leq 2[1 + (x + t \bar{x})^2 + (y + t \bar{y})^2 + (z + t \bar{z})^2][1 + (t \bar{x})^2 + (t \bar{y})^2 + (t \bar{z})^2]$$

$$= 2[1 + (x + t \bar{x})^2 + (y + t \bar{y})^2 + (z + t \bar{z})^2](1 + t^2)$$
By applying these observations, one may show, rather easily, that \( \gamma \) lies in \( W \). One may then verify that, in fact, \( \Gamma \) is bijective.

19° In turn, let \( L \) be the linear subspace of \( W \) defined by the following condition:
\[
\gamma \in L \quad \text{iff} \quad \gamma(0, x, y, z) = 0, \quad \gamma_t(0, x, y, z) = 0
\]
With reference to article 10°, we may presume to introduce a linear mapping \( \square \) carrying \( W \) to \( L \):
\[
\square \delta \equiv \gamma \quad (\delta \in W)
\]
defined in terms of the intermediate function \( \beta \) as follows:
\[
\beta(s, t, x, y, z) \equiv \frac{t}{4\pi t^2} \int \int \int \delta(s, x + t\bar{x}, y + t\bar{y}, z + t\bar{z})t^2 \cos(\theta)d\phi d\theta
\]
\[
\gamma(t, x, y, z) \equiv \int_0^t \beta(s, t - s, x, y, z)ds
\]
To justify the definition of \( \square \), we must show that \( \gamma \) lies in \( W \). It will follow, by design, that \( \gamma \) lies in \( L \) and that \( \square \gamma = \delta \). To that end, we need only apply the observations in the preceding article to show that the function:
\[
\alpha(s, t, x, y, z) \equiv \int \int \int \delta(s, x + t\bar{x}, y + t\bar{y}, z + t\bar{z}) \cos(\theta)d\phi d\theta
\]
defined on \( \mathbb{R}^5 \) is rapidly decreasing in \( x \), \( y \), and \( z \), locally uniformly in \( s \) and \( t \). Of course, we mean to say that, for any finite intervals \( U \) and \( V \) in \( \mathbb{R} \) and for any nonnegative integers \( p, k, \ell, a, b, \) and \( c \), the restriction of the function:
\[
(1 + x^2 + y^2 + z^2)^p \frac{\partial^{k+\ell+a+b+c}}{\partial s^k \partial t^\ell \partial x^a \partial y^b \partial z^c} \alpha(s, t, x, y, z)
\]
defined on \( \mathbb{R}^5 \) to the set \( U \times V \times \mathbb{R}^3 \) is bounded. Now one may show, rather easily, that \( \gamma \) lies in \( W \).

20° Let us emphasize that, in the current formal context, \( \square \) is a right inverse for \( \square \). That is:
\[
\square \square \delta = \delta \quad (\delta \in W)
\]
Moreover, the kernel \( K \) of \( \square \) and the range \( L \) of \( \square \) compose a direct sum decomposition of \( W \):
\[
W = K \oplus L
\]
21° At this point, we may summarize the properties of the Wave Operator in the following diagram:

[Diagram]

Retarded Potentials

22° Let us return to the particular solution of the Inhomogeneous Wave Equation defined in article 10° but let us modify the definition as follows:

\[(\star)\quad \gamma(t, x, y, z) \equiv \int_{-\infty}^{t} \beta(s, t - s, x, y, z)ds\]

For now, we ignore the question whether the foregoing integral is well defined. See articles 25° and 26°. By the computations in article 11°, we find that, once again, \(\gamma\) satisfies the Inhomogeneous Wave Equation:

\[(\circ)\quad \gamma_{tt}(t, x, y, z) - (\Delta \gamma)(t, x, y, z) = \delta(t, x, y, z)\]

However, it satisfies quite different Initial Conditions:

\[\gamma(0, x, y, z) = \int_{-\infty}^{0} \beta(s, -s, x, y, z)ds,\]
\[(\bullet)\quad \gamma_{t}(0, x, y, z) = \int_{-\infty}^{0} \beta_{t}(s, -s, x, y, z)ds\]
By a simple change of variables, we find that:

\[ 
\gamma(t, x, y, z) = \int_0^\infty \beta(t - s, s, x, y, z) ds 
= \int_0^\infty \left[ \frac{s}{4\pi s^2} \int \int_\Sigma \delta(t - s, x + s\bar{x}, y + s\bar{y}, z + s\bar{z}) s^2 \cos(\theta) d\phi d\theta \right] ds
\]

Let us convert Spherical Coordinates \((s\bar{x}, s\bar{y}, s\bar{z})\) to Cartesian Coordinates \((u, v, w)\):

\[ 
\begin{align*}
  u &\equiv x + s\bar{x} = x + s \cos(\theta) \cos(\phi) \\
v &\equiv y + s\bar{y} = y + s \cos(\theta) \sin(\phi) \\
w &\equiv z + s\bar{z} = z + s \sin(\theta)
\end{align*}
\]

We obtain:

\[ 
(*) \quad \gamma(t, x, y, z) = \frac{1}{4\pi} \int \int \int_{R^3} \frac{1}{s} \delta(t - s, u, v, w) du dv dw
\]

where:

\[ 
s \equiv \sqrt{(x - u)^2 + (y - v)^2 + (z - w)^2}
\]

Now we can provide an interpretation of the function \(\gamma\), just described.

23° To that end, we note that the Event \((t - s, u, v, w)\) occurs prior to the Event \((t, x, y, z)\), since \(t - s < t\). Moreover, the two are separated in Time and Space by a Null Interval:

\[ 
(t, x, y, z) - (t - s, u, v, w) = (s, x - u, y - v, z - w)
\]

since:

\[ 
s \equiv \sqrt{(x - u)^2 + (y - v)^2 + (z - w)^2}
\]

Hence, a light signal may pass from the former event to the latter, requiring \(s\) light seconds to do so. Now, for a given time \(t\), one calculates \(\gamma(t, x, y, z)\) at the position \((x, y, z)\) by:

1. considering an arbitrary position \((u, v, w)\)
2. calculating the travel time \(s\) from \((u, v, w)\) to \((x, y, z)\)
3. calculating \(\delta(t - s, u, v, w)\) at the retarded time \(t - s\)
4. finally, calculating the integral

One refers to \(\gamma\) as the Retarded Potential function for the Density function \(\delta\).
By a simple change of variables, we can present $\gamma$ in a different form, more convenient to computation:

\[
\gamma(t, x, y, z) = \frac{1}{4\pi} \int\int\int_{\mathbb{R}^3} \frac{1}{s} \delta(t - s, x - u, y - v, z - w)du dv dw
\]

where:

\[s \equiv \sqrt{u^2 + v^2 + w^2}\]

In this form for $\gamma$, the variable $s$ does not depend upon the variables $x$, $y$, and $z$. As a result, one can compute the partial derivatives of $\gamma$ easily.

**Rigour Redux (Incomplete)**

Let us examine the foregoing definition of Retarded Potentials. Given a Density function $\delta$ defined on $\mathbb{R}^4$, we defined the function $\beta$:

\[
\beta(s, t, x, y, z) \equiv \frac{t}{4\pi t^2} \int_\Sigma \delta(t - s, x - s \bar{x}, y + s \bar{y}, z + s \bar{z})t^2 cos(\theta) d\phi d\theta
\]

on $\mathbb{R}^5$ and the Retarded Potential function $\gamma$:

\[
\gamma(t, x, y, z) = \int_{-\infty}^{t} \beta(s, t - s, x, y, z) ds
\]

\[
= \int_{0}^{\infty} \beta(t - s, s, x, y, z) ds
\]

\[
= \int_{0}^{\infty} \frac{s}{4\pi s^2} \int_\Sigma \delta(t - s, x + s \bar{x}, y + s \bar{y}, z + s \bar{z})s^2 cos(\theta) d\phi d\theta ds
\]

\[
= \frac{1}{4\pi} \int\int\int_{\mathbb{R}^3} \frac{1}{s} \delta(t - s, u, v, w)du dv dw
\]

on $\mathbb{R}^4$, where:

\[u \equiv x + s \bar{x} = x + s \cos(\theta) \cos(\phi)\]

\[v \equiv y + s \bar{y} = y + s \cos(\theta) \sin(\phi)\]

\[w \equiv z + s \bar{z} = z + s \sin(\theta)\]

and:

\[s \equiv \sqrt{(x - u)^2 + (y - v)^2 + (z - w)^2}\]

In turn:

\[
\gamma(t, x, y, z) = \frac{1}{4\pi} \int\int\int_{\mathbb{R}^3} \frac{1}{s} \delta(t - s, x - u, y - v, z - w)du dv dw
\]

where:

\[s = \sqrt{u^2 + v^2 + w^2}\]
Of the five integrals which figure in the definition of $\gamma$, we may say that if one is well defined then, by transformation of variables, they are all well defined and mutually equal. However, we can readily exhibit an instance of a function $\delta$ in $W$ for which none of the integrals is well defined:

$$\delta(t, x, y, z) \equiv \ldots$$

26° Let $W_0$ be the linear subspace of $W$ consisting of all density functions $\delta$ such that the retarded potential function $\gamma$ is well defined. ............