Basic Definitions

1° Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. Let $\mathcal{E}$ denote the borel algebra of all measurable subsets of $\mathbb{R}$ and let $\mathcal{P}$ denote the convex set of all probability measures defined on $\mathcal{E}$.

2° We begin with the primitive idea of a physical system and with the primitive ideas of state and observable. For such a system, we introduce the sets $\mathcal{S}$ of all states and $\mathcal{O}$ of all observables and we introduce a mapping $\Pi$ carrying $\mathcal{S} \times \mathcal{O}$ to $\mathcal{P}$:

$$\Pi : \mathcal{S} \times \mathcal{O} \rightarrow \mathcal{P}$$

We refer to the ordered triple:

$$T = (\mathcal{S}, \mathcal{O}, \Pi)$$

as a physical theory for the given physical system. For any $S$ in $\mathcal{S}$, $A$ in $\mathcal{O}$, and $E$ in $\mathcal{E}$, we interpret:

$$\Pi(S, A)(E)$$

to be the probability that preparation of the physical system in the state $S$ and measurement of the observable $A$ yields a value in the set $E$.

Natural Requirements

3° For any physical theory $T$, we require that states and observables which are in practice indistinguishable are in fact identical, that is, for any $S_1$ and $S_2$ in $\mathcal{S}$:

$$[(\forall A \in \mathcal{O})(\Pi(S_1, A) = \Pi(S_2, A))] \Rightarrow [S_1 = S_2]$$

and, for any $A_1$ and $A_2$ in $\mathcal{O}$:

$$[(\forall S \in \mathcal{S})(\Pi(S, A_1) = \Pi(S, A_2))] \Rightarrow [A_1 = A_2]$$

Should these requirements fail, we would simply replace $\mathcal{S}$ and $\mathcal{O}$ by appropriate sets of equivalence classes.
The Functional Calculus

4° We also require that, for any real valued borel function \( f \) defined on \( \mathbb{R} \) and for any \( A \) in \( \mathcal{O} \), there is some \( B \) in \( \mathcal{O} \) such that:

\[
(\bullet) \quad (\forall S \in \mathcal{S})[\Pi(S, B) = f_*(\Pi(S, A))]
\]

Obviously, \( f \) and \( A \) uniquely determine \( B \). We say that \( B \) is a function of \( A \) and we denote \( B \) by \( f(A) \). By definition, for each \( E \) in \( \mathcal{E} \):

\[
\Pi(S, f(A))(E) = f_*(\Pi(S, A))(E) = \Pi(S, A)(f^{-1}(E))
\]

Commeasurability

5° In terms of the foregoing action of functions on observables, we can define the relation of commeasurability. Thus, for any observables \( B_1 \) and \( B_2 \) in \( \mathcal{O} \), we say that \( B_1 \) and \( B_2 \) are commeasurable iff there exists an observable \( A \) in \( \mathcal{O} \) such that both \( B_1 \) and \( B_2 \) are functions of \( A \).

6° Let \( \mathcal{O}_o \) be any subset of \( \mathcal{O} \). We say that the elements of \( \mathcal{O}_o \) are mutually commeasurable iff, for any \( B_1 \) and \( B_2 \) in \( \mathcal{O}_o \), \( B_1 \) and \( B_2 \) are commeasurable. We require that:

\[
(\bullet) \quad \text{for any subset } \mathcal{O}_o \text{ of } \mathcal{O}, \text{ if the elements of } \mathcal{O}_o \text{ are mutually commeasurable then there is some } A \text{ in } \mathcal{O} \text{ such that, for each } B \text{ in } \mathcal{O}_o, B \text{ is a function of } A
\]

We may refer to \( A \) as an \( ur \)-observable for \( \mathcal{O}_o \).

Partial Algebras

7° Let us describe the concept of a partial algebra. Let \( \mathcal{O} \) be an arbitrary set. We say that \( \mathcal{O} \) is a partial algebra iff we have supplied \( \mathcal{O} \) with a family \( \mathcal{A} \) of subsets of \( \mathcal{O} \) such that:

\[
(\circ) \quad \mathcal{O} = \cup \mathcal{A}
\]

\[
(\circ) \quad \text{for each } \mathcal{A} \text{ in } \mathcal{A}, \mathcal{A} \text{ is a commutative algebra over } \mathbb{R}
\]

\[
(\circ) \quad \text{for any } \mathcal{A}_1 \text{ and } \mathcal{A}_2 \text{ in } \mathcal{A}, \mathcal{A}_1 \cap \mathcal{A}_2 \text{ is itself in } \mathcal{A} \text{ and is a subalgebra of both } \mathcal{A}_1 \text{ and } \mathcal{A}_2
\]

\[
(\circ) \quad \text{for any subset } \mathcal{O}_o \text{ of } \mathcal{O}, \text{ if the elements of } \mathcal{O}_o \text{ are mutually compatible then there is some } \mathcal{A} \text{ in } \mathcal{A} \text{ such that } \mathcal{O}_o \subseteq \mathcal{A}
\]
To support the last of the foregoing conditions, we provide the following definitions. For any $B_1$ and $B_2$ in $O$, we say that $B_1$ and $B_2$ are compatible iff there is some $A$ in $A$ such that both $B_1$ and $B_2$ belong to $A$. In turn, we say that the elements of $O_o$ are mutually compatible iff, for any $B_1$ and $B_2$ in $O_o$, $B_1$ and $B_2$ are compatible.

8° Let us emphasize that, under the foregoing conditions, the neutral elements 0 and 1 for the various algebras $A$ in $A$ are all the same.

**Homomorphisms of Partial Algebras**

9° Let $O'$ and $O''$ be partial algebras and let $A'$ and $A''$ be the corresponding families of commutative algebras over $R$. Let $H$ be a mapping carrying $O'$ to $O''$. We refer to $H$ as a homomorphism iff, for any $A'$ in $A'$, there is some $A''$ in $A''$ such that $H(A') \subseteq A''$ and such that the restriction/contraction of $H$ to $A'$ and $A''$ is (in the usual sense) a homomorphism.

**The Partial Algebra of Observables**

10° Let us return to the context of the physical theory $T$. Now we simply declare that:

- (●) the set $O$ of observables is a partial algebra

As required, we mention the corresponding family $A$ of commutative algebras over $R$. Naturally, we impose a condition which intertwines the structure of $O$ just defined with the foregoing functional calculus:

- (●) for any $B_1$ and $B_2$ in $O$, $B_1$ and $B_2$ are compatible iff they are commeasurable, in which case, for any $A$ in $O$ and for any real valued borel functions $f_1$ and $f_2$ defined on $R$:

\[
(B_1 = f_1(A)) \land (B_2 = f_2(A)) \implies (B_1 + B_2 = (f_1 + f_2)(A)) \land (B_1 B_2 = (f_1 f_2)(A))
\]

11° Let $O_o$ be any subset of $O$. Obviously, the elements of $O_o$ are mutually compatible iff the elements of $O_o$ are mutually commeasurable. In such a context, we may introduce an ur-observable $A$ for $O_o$. By elementary argument, we would find that the elements of $O_o \cup \{A\}$ are mutually compatible. Hence, there would be some $A$ in $A$ such that $O_o \cup \{A\} \subseteq A$.

12° Let us introduce certain innocuous but useful conditions on $A$. First, let $A_1$ and $A_2$ be any commutative algebras over $R$ such that $A_1$ is a subalgebra of $A_2$. We assume that:
if $A_2 \in A$ then $A_1 \in A$

Second, let $A_o$ be any chain in $A$. That is, let $A_o$ be any subset of $A$ such that, for any $A_1$ and $A_2$ in $A_o$, either $A_1$ is a subalgebra of $A_2$ or $A_2$ is a subalgebra of $A_1$. Naturally, $\cup A_o$ is a commutative algebra over $R$. We assume that:

- $\cup A_o \in A$

13° Under the second of the foregoing conditions, we may apply Zorn’s Lemma to infer that, for any $A$ in $A$, there is some $M$ in $A$ such that $A \subseteq M$ and such that $M$ is maximal. The latter assertion means that, for any $B$ in $A$, if $M \subseteq B$ then $M = B$.

14° We shall refer to a maximal member of $A$ as a context.

Boolean Rings

15° Let us review the basic properties of boolean rings. Let $B$ be any ring. We say that $B$ is a boolean ring iff, for each $X$ in $B$, $X^2 = X$. Let $B$ be such a ring. Let us represent the operation of addition not by $+$ but by $\oplus$. We find that, for any $Y$ in $B$:

$$Y \oplus Y = (Y \oplus Y)^2 = Y^2 \oplus Y^2 \oplus Y^2 \oplus Y^2 = Y \oplus Y \oplus Y \oplus Y$$

so that, $Y \oplus Y = 0$. In turn, for any $Y_1$ and $Y_2$ in $B$:

$$Y_1 \oplus Y_2 = (Y_1 \oplus Y_2)^2 = Y_1^2 \oplus Y_1 Y_2 \oplus Y_2 Y_1 \oplus Y_2^2 = Y_1 \oplus Y_1 Y_2 \oplus Y_2 Y_1 \oplus Y_2$$

so that, $Y_1 Y_2 \oplus Y_2 Y_1 = 0$. Hence, $Y_1 Y_2 = Y_2 Y_1$. Consequently, boolean rings must be commutative.

16° Let $A$ be a commutative algebra over $R$. Let $B$ be the subset of $A$ consisting of all idempotent elements of $A$, that is, the subset consisting of all elements $X$ for which $X^2 = X$. Clearly, $B$ is closed under multiplication in $A$. Let us supply $B$ with the operation of multiplication which descends from $A$. However, $B$ is not (in general) closed under addition in $A$. In compensation, let us supply $B$ with the operation of addition defined as follows:

$$X \oplus Y = X + Y - 2XY$$

where $X$ and $Y$ are any elements of $B$. Remarkably, under the operations of addition and multiplication just described, $B$ is a boolean ring. In future, we will refer to $B$ as the boolean “subring” of $A$, composed of the idempotent elements of $A$. 

4
Let $B$ be a boolean ring. Let 0 and 1 be the neutral elements for $B$. We introduce the relation $\leq$ on $B$ as follows:

$$X_1 \leq X_2 \iff X_1 = X_1X_2$$

One can easily check that $\leq$ is a partial order relation on $B$. Obviously, for each $X$ in $B$, $0 \leq X \leq 1$. Moreover, for any $Y_1$ and $Y_2$ in $B$:

$$Y_1 \land Y_2 = Y_1Y_2 \quad \text{and} \quad Y_1 \lor Y_2 = Y_1 \oplus Y_2 \oplus Y_1Y_2$$

serve as the infimum and the supremum, respectively, of the set:

$$\{Y_1, Y_2\}$$

That is, $Y_1 \land Y_2 \leq Y_1$ and $Y_1 \land Y_2 \leq Y_2$, while, for any $X$ in $B$, if $X \leq Y_1$ and $X \leq Y_2$ then $X \leq Y_1 \land Y_2$. Moreover, $Y_1 \leq Y_1 \lor Y_2$ and $Y_2 \leq Y_1 \lor Y_2$, while, for any $Z$ in $B$, if $Y_1 \leq Z$ and $Y_2 \leq Z$ then $Y_1 \lor Y_2 \leq Z$.

Finally, for each $X$ in $B$, we define the complement of $X$ as follows:

$$X' = 1 \oplus X$$

Clearly:

$$X \land X' = 0, \quad X \lor X' = 1, \quad X'' = X$$

We find that, for any $X_1$ and $X_2$ in $B$:

$$X_1 \leq X_2 \iff X_2' \leq X_1'$$

We say that $B$ is complete iff, for each subset $C$ of $B$, there are elements:

$$\land C \quad \text{and} \quad \lor C$$

of $B$ which serve as the infimum and supremum of $C$, respectively. That is:

$$(\forall Y \in C)(\land C \leq Y) \land (\forall X \in B)[(\forall Y \in C)(X \leq Y) \implies (X \leq \land C)]$$

and:

$$(\forall Y \in C)(Y \leq \lor C) \land (\forall Z \in B)[(\forall Y \in C)(Y \leq Z) \implies (\lor C \leq Z)]$$

We say that $B$ is countably complete iff, for each countable subset $C$ of $B$, there are elements:

$$\land C \quad \text{and} \quad \lor C$$
of $\mathcal{B}$ which serve as the infimum and supremum of $\mathcal{C}$, respectively. Of course, in this case, we may display the elements of $\mathcal{C}$ in a list:

$$Y_1, Y_2, Y_3, Y_4, \ldots$$

and we may choose to denote the infimum and the supremum of $\mathcal{C}$ as follows:

$$\bigwedge \mathcal{C} = \bigwedge_j Y_j, \quad \bigvee \mathcal{C} = \bigvee_j Y_j$$

21° For any $X_1$ and $X_2$ in $\mathcal{B}$, we say that $X_1$ and $X_2$ are disjoint iff:

$$X_1 \land X_2 = 0$$

It is the same to say that $X_1 \leq X_2'$ or that $X_2 \leq X'_1$. For any subset $\mathcal{C}$ of $\mathcal{B}$, we say that the elements of $\mathcal{C}$ are mutually disjoint iff, for any $Y_1$ and $Y_2$ in $\mathcal{C}$, $Y_1$ and $Y_2$ are disjoint.

22° We say that $\mathcal{B}$ is countably generated iff, for each subset $\mathcal{C}$ of $\mathcal{B}$, if the elements of $\mathcal{C}$ are mutually disjoint then $\mathcal{C}$ is countable.

23° One can easily show that if $\mathcal{B}$ is countably generated and countably complete then $\mathcal{B}$ is complete.

24° Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be boolean rings. Let $H$ be a homomorphism carrying $\mathcal{B}_1$ to $\mathcal{B}_2$. For any $X$ and $Y$ in $\mathcal{B}_1$, we find that:

$$X \leq Y \iff X = XY \implies H(X) = H(X)H(Y) \iff H(X) \leq H(Y)$$

Hence, $H$ preserves order.

**Partial Boolean Rings**

25° Let us describe the concept of a partial boolean ring. Let $\mathcal{Q}$ be an arbitrary set. We say that $\mathcal{Q}$ is a partial boolean ring iff we have supplied $\mathcal{Q}$ with a family $\mathcal{B}$ of subsets of $\mathcal{Q}$ such that:

- $(\circ)$ $\mathcal{Q} = \cup \mathcal{B}$
- $(\circ)$ for each $\mathcal{B}$ in $\mathcal{B}$, $\mathcal{B}$ is a boolean ring
- $(\circ)$ for any $\mathcal{B}_1$ and $\mathcal{B}_2$ in $\mathcal{B}$, $\mathcal{B}_1 \cap \mathcal{B}_2$ is itself in $\mathcal{B}$ and is a boolean subring of both $\mathcal{B}_1$ and $\mathcal{B}_2$
- $(\circ)$ for any subset $\mathcal{Q}_0$ of $\mathcal{Q}$, if the elements of $\mathcal{Q}_0$ are mutually compatible then there is some $\mathcal{B}$ in $\mathcal{B}$ such that $\mathcal{Q}_0 \subseteq \mathcal{B}$
To support the last of the foregoing conditions, we provide the following definitions. For any \( Q_1 \) and \( Q_2 \) in \( Q \), we say that \( Q_1 \) and \( Q_2 \) are compatible iff there is some \( B \) in \( B \) such that both \( Q_1 \) and \( Q_2 \) belong to \( B \). In turn, we say that the elements of \( Q_o \) are mutually compatible iff, for any \( Q_1 \) and \( Q_2 \) in \( Q_o \), \( Q_1 \) and \( Q_2 \) are compatible.

26° Let us emphasize that, under the foregoing conditions, the neutral elements 0 and 1 for the various boolean rings \( B \) in \( B \) are all the same.

27° We say that the partial boolean ring \( Q \) is complete iff, for each \( B' \) in \( B \), there is some \( B'' \) in \( B \) such that \( B' \subseteq B'' \) and such that \( B'' \) is complete. In this context, we mean to require that, for any subset \( C \) of \( B_1 \), if there are elements \( \land_1 C \) and \( \lor_1 C \) in \( B_1 \) which serve, respectively, as the infimum and the supremum of \( C \) in \( B_1 \) then \( \land_1 C = \land_2 C \) and \( \lor_1 C = \lor_2 C \), where \( \land_2 C \) and \( \lor_2 C \) are the elements in \( B_2 \) which serve, respectively, as the infimum and the supremum of \( C \) in \( B_2 \).

**Homomorphisms of Partial Boolean Rings**

28° Let \( Q' \) and \( Q'' \) be partial boolean rings and let \( B' \) and \( B'' \) be the corresponding families of boolean rings. Let \( H \) be a mapping carrying \( Q' \) to \( Q'' \). We refer to \( H \) as a homomorphism iff, for any \( B' \) in \( B' \), there is some \( B'' \) in \( B'' \) such that \( H(B') \subseteq B'' \) and such that the restriction/contraction of \( H \) to \( B' \) and \( B'' \) is (in the usual sense) a homomorphism.

**Questions**

29° Let us return to the context of the physical theory \( T \). Let \( Q \) be any observable in \( O \). We contend that \( Q^2 = Q \) iff:

\[
(\forall S \in S)[\Pi(S, Q)(\{0, 1\}) = 1]
\]

To prove the contention, we introduce the real valued borel function \( \sigma \) defined on \( R \) as follows: for each \( x \) in \( R \), \( \sigma(x) = x^2 \). By article 10°, \( Q^2 = \sigma(Q) \). Let us assume that condition (\* \( ) holds. Let \( S \) be any state in \( S \). Clearly:

\[
\Pi(S, Q^2)(\{0\}) = \Pi(S, Q)(\sigma^{-1}(\{0\})) = \Pi(S, Q)(\{0\})
\]

Moreover, since \( \Pi(S, Q)(\{-1\}) = 0 \):

\[
\Pi(S, Q^2)(\{1\}) = \Pi(S, Q)(\sigma^{-1}(\{1\})) = \Pi(S, Q)(\{-1, 1\}) = \Pi(S, Q)(\{1\})
\]

Obviously:

\[
\Pi(S, Q^2)(\{0, 1\}) = \Pi(S, Q)(\{0, 1\}) = 1
\]
We infer that:

\[ \Pi(S, Q^2)(R\setminus\{0, 1\}) = 0 = \Pi(S, Q)(R\setminus\{0, 1\}) \]

By article 3°, we infer that \( Q^2 = Q \). Now let us assume that \( Q^2 = Q \). Let \( S \) be any state in \( S \). Let \( E \) be any (borel) set in \( E \). Clearly:

\[ (\star) \quad \Pi(S, Q)(E) = \Pi(S, Q^2)(E) = \Pi(S, Q)(\sigma^{-1}(E)) \]

Let \( R^- \) be the (borel) subset of \( R \) consisting of all negative real numbers. Obviously, \( \sigma^{-1}(R^-) = \emptyset \). Hence, by relation (\( \star \)), \( \Pi(S, Q)(R^-) = 0 \). Let \( x \) be any positive real number and let \( y = \sigma(x) \). If \( x < 1 \) then, by relation (\( \star \)), \( \Pi(S, Q)((y, x)) = 0 \). If \( 1 < x \) then, by relation (\( \star \)), \( \Pi(S, Q)((x, y)) = 0 \). Now, by elementary steps, we find that:

\[ \Pi(S, Q)((0, 1)) = 0 \quad \text{and} \quad \Pi(S, Q)((1, \rightarrow)) = 0 \]

Hence, \( \Pi(S, Q)(\{0, 1\}) = 1 \). We infer that condition (\( \star \)) holds.

30° Now let \( Q \) be the subset of \( O \) consisting of all observables \( Q \) such that \( Q^2 = Q \). We refer to such observables as questions. For any \( Q \) in \( Q \) and \( S \) in \( S \), we interpret:

\[ \Pi(S, Q)(\{0\}) \quad \text{and} \quad \Pi(S, Q)(\{1\}) \]

to be the probabilities that preparation of the physical system in the state \( S \) and “measurement” of the question \( Q \) will yield the answers “no” and “yes,” respectively.

31° Questions are legion. Indeed, let \( A \) be any observable in \( O \), let \( F \) be any borel set in \( E \), and let \( ch_F \) be the characteristic function of \( F \):

\[ ch_F(x) = \begin{cases} 0 & \text{if } x \notin F \\ 1 & \text{if } x \in F \end{cases} \]

Obviously, \( ch_F^2 = ch_F \). By article 10°, it is plain that \( ch_F(A) \) is a question in \( Q \).

32° Now let \( f \) be a real valued borel function defined on \( R \), let \( F = f^{-1}(\{1\}) \), and let \( g = ch_F \). We contend that if \( f(A) \) is a question then \( g(A) = f(A) \). To prove the contention, we note that, for each \( S \) in \( S \):

\[ \Pi(S, g(A))(\{1\}) = \Pi(S, A)(F) = \Pi(S, f(A))(\{1\}) \]

and that:

\[ \Pi(S, g(A))(\{0\}) = 1 - \Pi(S, A)(F) = \Pi(S, f(A))(\{0\}) \]

We infer that:

\[ \Pi(S, g(A))(R\setminus\{0, 1\}) = 0 = \Pi(S, f(A))(R\setminus\{0, 1\}) \]

By article 3°, we infer that \( g(A) = f(A) \).
**LOGIC: the Partial Boolean Ring of Questions**

33° Let us recall that $O$ is a partial algebra and let us recover the family $A$ of commutative algebras over $R$ with which $O$ is supplied. Let $B$ be the corresponding family of boolean rings, defined as follows:

$$B = Q \cap A$$

We mean to say that, for any subset $B$ of $Q$, $B \in B$ iff there is some $A$ in $A$ such that $B = Q \cap A$. Of course, $B$ is the boolean “subring” of $A$, composed of the idempotent elements of $A$. Obviously:

- the set $Q$ of questions is a partial boolean ring

We refer to $Q$ as the **LOGIC** for the physical theory $T$.

34° Let $Q_1$ and $Q_2$ be compatible questions in $Q$. We contend that $Q_1 \leq Q_2$ iff:

$$(*): (\forall S \in S) [\Pi(S, Q_1)(\{1\}) \leq \Pi(S, Q_2)(\{1\})]$$

To prove the contention, we argue as follows. By article 34°, we may introduce an observable $B$ in $O$ and (borel) sets $F_1$ and $F_2$ in $E$ such that:

$$Q_1 = ch_{F_1}(B), \quad Q_2 = ch_{F_2}(B)$$

Let us assume that $Q_1 \leq Q_2$. By definition, $Q_1 = Q_1 Q_2$. Consequently:

$$Q_1 = ch_{F_1 \cap F_2}(B)$$

Accordingly, we may assume that $F_1 = F_1 \cap F_2 \subseteq F_2$. Hence, for any state $S$ in $S$:

$$\Pi(S, Q_1)(\{1\}) = \Pi(S, B)(F_1) \leq \Pi(S, B)(F_2) = \Pi(S, Q_2)(\{1\})$$

We infer that condition $(*)$ holds.

35° Now let us assume that condition $(*)$ holds. We claim that $Q_1 \leq Q_2$. To support the claim, we impose the following (more or less natural) condition on the logic $Q$:

- $(\forall Q \in Q)[(Q \neq 0) \implies (\exists S \in S)(\Pi(S, Q)(\{1\}) = 1)]$

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9
The Convex Set of States

36° We also declare that:

(•) the set $S$ of states is countably convex

By this condition, we mean that, for any countable family:

$$S_1, S_2, S_3, \ldots$$

in $S$ and for a corresponding family:

$$c_1, c_2, c_3, \ldots$$

of nonnegative real numbers, if:

$$\sum_j c_j = 1$$

then there is some $S$ in $S$ such that, for each $A$ in $\mathcal{O}$ and for each $E$ in $\mathcal{E}$:

$$\Pi(S, A)(E) = \sum_j c_j \Pi(S_j, A)(E)$$

By article 3°, $S$ would be unique. We express $S$ as a convex sum:

$$S = \sum_j c_j S_j$$

37° Let us recall that, for any $S$ in $\mathcal{S}$, $S$ is an extreme point of $\mathcal{S}$ iff, for any $S_1$ and $S_2$ in $\mathcal{S}$ and for any nonnegative real numbers $c_1$ and $c_2$:

$$(c_1 + c_2 = 1) \land (S = c_1 S_1 + c_2 S_2) \implies (S = S_1) \lor (S = S_2)$$

We refer to the extreme points in $\mathcal{S}$ as pure states.

Reconstruction of $\mathcal{S}$, $\mathcal{O}$, and $\Pi$ from $Q$

38° For any $S$ in $\mathcal{S}$, we introduce the mapping:

$$\bar{S} : Q \longrightarrow [0, 1]$$

as follows:

$$\bar{S}(Q) = \Pi(S, Q)(\{1\})$$
where $Q$ is any question in $Q$. ..... Obviously, $\bar{S}(0) = 0$ and $\bar{S}(1) = 1$. ..... Moreover, for each $Q$ in $Q$:

$$\bar{S}(Q') = 1 - \bar{S}(Q)$$

...... Finally, we contend that, for each countable subset:

$$Q_1, Q_2, Q_3, Q_4, \ldots$$

of $Q$, if the elements are mutually compatible and mutually disjoint then:

$$\bar{S}(\bigvee_j Q_j) = \sum_j \bar{S}(Q_j)$$

......

39° Under these conditions, we refer to $\bar{S}$ as a normalized measure on $Q$.

40° One can easily check that, for any $S_1$ and $S_2$ in $S$, if $\bar{S}_1 = \bar{S}_2$ then $S_1 = S_2$.

41° For any $A$ in $A$, we introduce the mapping:

$$\bar{A} : E \rightarrow Q$$

as follows:

$$\bar{A}(E) = ch_E(A)$$

where $E$ is any (borel) set in $E$. We contend that, for each countable subset:

$$E_1, E_2, E_3, E_4, \ldots$$

of $E$, if the sets are mutually disjoint then the elements:

$$\bar{A}(E_1), \bar{A}(E_2), \bar{A}(E_3), \bar{A}(E_4), \ldots$$

in $Q$ are mutually compatible and mutually disjoint.

......

42° Under these conditions, we refer to $\bar{A}$ as a question-valued measure defined on $E$.

43° One can easily check that, for any $A_1$ and $A_2$ in $A$, if $\bar{A}_1 = \bar{A}_2$ then $A_1 = A_2$.

44° Relate $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_1 + \bar{A}_2$, and $\bar{A}_1 \bar{A}_2$. 

11
45° Obviously, for any $S$ in $\mathcal{S}$ and for any $A$ in $\mathcal{O}$, the following relation is both meaningful and true:

$$\Pi(S, A) = \tilde{S} \cdot \tilde{A}$$

because, for any (borel) set $E$ in $\mathcal{E}$:

$$\tilde{S}(\tilde{A}(E)) = \tilde{S}(\tilde{A}(E)) = \tilde{S}(\tilde{A}(E)) = \Pi(S, \tilde{A}(E))$$

46° At this point, we might say that the basic structure for a physical theory $T$ is the underlying logic $Q$ and that the structures $\mathcal{S}$, $\mathcal{O}$, and $\Pi$ can be reconstructed from $Q$.

47° States are positive linear functionals on the “bounded” observables.

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**Homomorphisms of Physical Theories**

48°

49°

50°

**Classical Physical Theories**

51° For a *classical physical theory*:

$$T = (\mathcal{S}, \mathcal{O}, \Pi)$$

we begin with a standard borel space $X$. The logic $Q$ of questions is the boolean ring composed of all borel subsets $Q$ of $X$. The states in $\mathcal{S}$ are the probability measures $S$ defined on $Q$; the observables in $\mathcal{O}$ are the real valued borel functions $A$ defined on $X$; and:

$$\Pi(S, A) = \tilde{S} \cdot \tilde{A}$$

To be clear, let us note that $\tilde{S} = S$ and that, for any $E$ in $\mathcal{E}$:

$$\tilde{A}(E) = A^{-1}(E) \quad \text{and} \quad \Pi(S, A)(E) = S(A^{-1}(E))$$
so that:

\[ \Pi(S, A) = A_x(S) \]

52° The pure states in \( S \) are the probability measures of the form:

\[ \Delta_x \]

where \( x \) is any point in \( X \). By definition:

\[ \Delta_x(Q) = \begin{cases} 
0 & \text{if } x \notin Q \\
1 & \text{if } x \in Q 
\end{cases} \quad (Q \in \mathcal{Q}) \]

Clearly, for any \( A \) in \( \mathcal{A} \), the mean \( m \) of \( \Pi(\Delta_x, A) \) is \( A(x) \):

\[ m = \int_{\mathbb{R}} a \Pi(\Delta_x, A)(da) = \int_{\mathbb{R}} a \Delta_{A(x)}(da) = A(x) \]

and the standard deviation \( s \) is 0:

\[ s^2 = \int_{\mathbb{R}} (a - m)^2 \Pi(\Delta_x, A)(da) = \int_{\mathbb{R}} (a - m)^2 \Delta_{A(x)}(da) = 0 \]

Quantum Physical Theories

53° For a quantum physical theory:

\[ T = (S, \mathcal{O}, \Pi) \]

we begin with a separable complex hilbert space \( H \). For any \( \psi_1 \) and \( \psi_2 \) in \( H \), we represent the inner product of \( \psi_1 \) and \( \psi_2 \) as follows:

\[ \langle \psi_1, \psi_2 \rangle \]

The logic \( \mathcal{Q} \) of questions is the partial boolean ring composed of all self adjoint projection operators \( Q \) on \( H \). Such operators are coextensive with closed linear subspaces \( \tilde{Q} \) of \( H \):

\[ \tilde{Q} = \text{ran}(Q) \]

The states in \( S \) are the normalized nonnegative self adjoint operators of trace class on \( H \). One refers to such an operator as a density operator on \( H \). By the Theorem of A. M. Gleason, density operators \( S \) are coextensive with normalized measures \( \bar{S} \) on \( \mathcal{Q} \):

\[ \bar{S}(Q) = \text{tr}(SQ) \]
where \( Q \) is any question in \( Q \). The observables in \( \mathcal{O} \) are the (not necessarily bounded but in any case densely defined) self adjoint operators \( A \) on \( \mathcal{H} \). By the Theorem of M. H. Stone, such observables are coextensive with projection-valued measures \( \bar{A} \) on \( \mathcal{E} \):

\[
\langle A(\psi_1), \psi_2 \rangle = \int_{\mathbb{R}} a \langle \bar{A}(da)(\psi_1), \psi_2 \rangle
\]

where \( \psi_1 \) and \( \psi_2 \) are any vectors in \( \mathcal{H} \) and where \( \psi_1 \in \text{dom}(A) \). Finally:

\[
\Pi(S, A) := S \cdot \bar{A}
\]

so that, for any \( E \) in \( \mathcal{E} \):

\[
\Pi(S, A)(E) = \text{tr}(S\bar{A}(E))
\]

For each unit vector \( \varphi \) in \( \mathcal{H} \), one forms the self adjoint projection operator \( R_\varphi \) as follows:

\[
R_\varphi(\psi) = \langle \psi, \varphi \rangle \varphi
\]

where \( \psi \) is any vector in \( \mathcal{H} \). Obviously:

\[
\text{ran}(R_\varphi) = C\varphi
\]

so that \( \text{ran}(R_\varphi) \) is 1-dimensional. As noted, one can identify such operators with their ranges:

\[
\bar{R}_\varphi = \text{ran}(R_\varphi)
\]

Now one can regard \( R_\varphi \) either as a state or as a question:

\[
S_\varphi = R_\varphi = Q_\varphi
\]

Under the first view, one obtains precisely the pure states in \( S \). Under the second view, one interprets \( Q_\varphi \) to be the question whether the physical system is in the pure state \( S_\varphi \). Let us explain this interpretation. For any unit vectors \( \varphi_1 \) and \( \varphi_2 \) in \( \mathcal{H} \):

\[
\Pi(S_{\varphi_1}, Q_{\varphi_2})(\{1\}) = \text{tr}(S_{\varphi_1}Q_{\varphi_2})
\]

\[
= \langle (S_{\varphi_1}Q_{\varphi_2})(\varphi_2), \varphi_2 \rangle
\]

\[
= \langle \varphi_2, \varphi_1 \rangle \varphi_1, \varphi_2 \rangle
\]

\[
= |\langle \varphi_1, \varphi_2 \rangle|^2
\]

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Of course, \( \Pi(S_{\varphi_1}, Q_{\varphi_2})(\{1\}) \) is the probability that preparation of the physical system in the pure state \( S_{\varphi_1} \) and “measurement” of the question \( Q_{\varphi_2} \) will yield the answer “yes.” Clearly:

\[
\Pi(S_{\varphi_1}, Q_{\varphi_2})(\{1\}) = 1 \quad \text{iff} \quad |\langle \varphi_1, \varphi_2 \rangle|^2 = 1 \\
\text{iff} \quad (\exists z \in \mathbb{C})[|z = 1| \land (\varphi_2 = z\varphi_1)] \\
\text{iff} \quad S_{\varphi_1} = Q_{\varphi_2}
\]

These observations “justify” the foregoing interpretation of \( Q_{\varphi} \). One refers to the numbers:

\[
|\langle \varphi_1, \varphi_2 \rangle|^2
\]

as \textit{transition probabilities}. Such numbers are the fundamental measurable quantities for a quantum theory.

55° For each \( S \) in \( \mathcal{S} \), one can introduce a countable family:

\( \varphi_1, \varphi_2, \varphi_3, \varphi_4, \ldots \)

of mutually orthogonal unit vectors in \( \mathcal{H} \) and a corresponding family:

\( w_1, w_2, w_3, w_4, \ldots \)

of nonnegative real numbers such that:

\[
\sum_j w_j = 1 \quad \text{and} \quad S = \sum_j w_j S_{\varphi_j}
\]

We intend that the foregoing series converge strongly. For any \( A \) in \( \mathcal{O} \) and \( E \) in \( \mathcal{E} \):

\[
\Pi(S, A)(E) = tr(S\tilde{A}(E)) \\
= \sum_j w_j tr(S_{\varphi_j} \tilde{A}(E)) \\
= \sum_j w_j \langle \tilde{A}(E)(\varphi_j), \varphi_j \rangle
\]

and:

\[
\Pi(S, A)(E) = \sum_j w_j \Pi(S_j, A)(E)
\]

Consequently, as the notation suggests, \( S \) is a countable convex sum of pure states.
56° For each unit vector $\varphi$ in $\mathbf{H}$:

$$\varphi \in \text{dom}(A) \iff \int_{\mathbb{R}} a^2 \langle \bar{A}(da), \varphi \rangle < \infty$$

For the corresponding pure state $S_{\varphi}$, one can compute the mean $m$ and the standard deviation $s$ for $\Pi(S_{\varphi}, A)$ as follows:

$$m = \int_{\mathbb{R}} a \Pi(S_{\varphi}, A)(da)$$
$$= \int_{\mathbb{R}} a \langle \bar{A}(da), \varphi \rangle$$
$$= \langle \bar{A}(\varphi), \varphi \rangle$$

and:

$$s^2 = \int_{\mathbb{R}} (a - m)^2 \Pi(S_{\varphi}, A)(da)$$
$$= \int_{\mathbb{R}} (a - m)^2 \langle \bar{A}(da), \varphi \rangle$$
$$= \langle (A - mI)^2(\varphi), \varphi \rangle$$

where $I$ is the identity operator on $\mathbf{H}$. In general, $s \neq 0$. However, if $\varphi$ is an eigenvector for $A$:

$$A(\varphi) = a\varphi$$

then $m = a$ and $s = 0$.

The Uncertainty Principle

57° Let us describe a special feature of the quantum physical theory $(\mathcal{S}, \mathcal{O}, \Pi)$. Let $\varphi$ be a unit vector in $\mathbf{H}$ and let $A_1$ and $A_2$ be self-adjoint operators on $\mathbf{H}$ which meet the following condition:

$$\varphi \in \text{dom}(A_1) \cap \text{dom}(A_2) \cap \text{dom}(A_1A_2) \cap \text{dom}(A_2A_1)$$

Let $m_1$ and $m_2$ be the means for $\Pi(S_{\varphi}, A_1)$ and $\Pi(S_{\varphi}, A_2)$ and let $\hat{A}_1$ and $\hat{A}_2$ be the self adjoint operators on $\mathbf{H}$, defined as follows:

$$\hat{A}_1 = A_1 - m_1 I, \quad \hat{A}_2 = A_2 - m_2 I$$

Let $s_1$ and $s_2$ be the standard deviations for $\Pi(S_{\varphi}, A_1)$ and $\Pi(S_{\varphi}, A_2)$. For each real number $a$:

$$0 \leq \langle (\hat{A}_1 + a \frac{1}{i}\hat{A}_2)(\varphi), (\hat{A}_1 + a \frac{1}{i}\hat{A}_2)(\varphi) \rangle$$
$$= \langle \hat{A}_1^2(\varphi), \varphi \rangle + a \langle \frac{1}{i}(\hat{A}_1\hat{A}_2 - \hat{A}_2\hat{A}_1)(\varphi), \varphi \rangle + a^2 \langle \hat{A}_2^2(\varphi), \varphi \rangle$$
$$= s_1^2 + a\xi + a^2 s_2^2$$

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where:
\[ \xi := \langle \frac{1}{i}(\hat{A}_1\hat{A}_2 - \hat{A}_2\hat{A}_1)(\varphi), \varphi \rangle \]
which is a real number. It follows that:
\[ \frac{1}{4}\xi^2 \leq \delta_1^2 \delta_2^2 \]
The relation just derived yields the Uncertainty Principle of Heisenberg. For instance, if:
\[ \frac{1}{i}(\hat{A}_1\hat{A}_2 - \hat{A}_2\hat{A}_1)(\varphi) = \varphi \]
(so that \( \xi = 1 \)) then:
\[ \frac{1}{2} \leq s_1 s_2 \]
Hence, the statistics of measurement for \( \Pi(S_\varphi, A_1) \) and \( \Pi(S_\varphi, A_2) \) will show a striking property: the more accurate the empirical estimate of \( m_1 \), the less accurate the empirical estimate of \( m_2 \); and conversely.

\textbf{Von Neumann, Bell}

58° Let \( T' \) be a quantum physical theory. Can we design a classical physical theory \( T'' \) and an injective homomorphism \( H \) carrying \( T' \) to \( T'' \)?

\textbf{Dynamics}

59° At this point, one might draw an analogy between our description of a physical theory:
\[ T = (S, O, \Pi) \]
and the composition of a play, for which there is stage and cast but no plot. To complete the description, we must now add to \( S, O, \) and \( \Pi \) the several features of \textit{dynamics}. 