01° Let $X$ be a measurable space, with $\sigma$-algebra $\mathcal{A}$. Let $\mu$ be a measure on $\mathcal{A}$. Let:

$$f_1 \leq f_2 \leq \ldots \leq f_j \leq \ldots$$

be an increasing sequence of measurable nonnegative extended real valued functions defined on $X$. Let $f$ be the corresponding pointwise limit:

$$f(x) = \lim_{j \to \infty} f_j(x)$$

where $x$ is any member of $X$. Of course, $f$ is measurable. We contend that:

$$\int_X f(x) \mu(dx) = \lim_{j \to \infty} \int_X f_j(x) \mu(dx)$$

This assertion is the substance of the Monotone Convergence Theorem. To prove the contention, we argue as follows. Clearly:

$$\int_X f_j(x) \mu(dx) \leq \int_X f_{j+1}(x) \mu(dx) \leq \ldots \leq \int_X f(x) \mu(dx) \quad (j \in \mathbb{Z}^+)$$

Let:

$$s \equiv \lim_{j \to \infty} \int_X f_j(x) \mu(dx)$$

Clearly:

$$s \leq \int_X f(x) \mu(dx)$$

If $s = \infty$ then the contention is obvious. Let us assume that $s < \infty$. Let $h$ be any simple function defined on $X$ such that $h \leq f$. Let $c$ be any real number for which $0 < c < 1$. For each positive integer $j$, let $E_j$ be the set in $\mathcal{A}$ consisting of all members $x$ of $X$ such that $ch(x) \leq f_j(x)$. Clearly:

$$E_j \subseteq E_{j+1} \quad (j \in \mathbb{Z}^+) \quad \text{and} \quad \bigcup_{j=1}^{\infty} E_j = X$$

Consequently:

$$c \int_{E_j} h(x) \mu(dx) \leq \int_X f_j(x) \mu(dx) \leq s \quad (j \in \mathbb{Z}^+)$$

It follows that:

$$\int_X h(x) \mu(dx) \leq s$$

Hence:

$$\int_X f(x) \mu(dx) \leq s$$

The proof is complete.