HYPERBOLIC TRIANGLES

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1° We plan to describe the construction, by straightedge and compass, of certain geodesic triangles in the hyperbolic plane.

2° Let us begin by explaining the relevant terminology. First, the hyperbolic plane is the circular disk \( H \) in the Cartesian plane \( \mathbb{R}^2 \), composed of all points \((x, y)\) for which:

\[ x^2 + y^2 < 1 \]

Second, the hyperbolic lines in \( H \) are the intersections with \( H \) of circles in \( \mathbb{R}^2 \) which meet the boundary of \( H \) at right angles. We shall refer to such circles as hypercircles.

By elementary argument, one can prove that, for any two distinct points in \( H \), there is precisely one hypercircle \( C \) in \( \mathbb{R}^2 \) such that both points are contained in \( C \).

3° The geodesic arcs in \( H \) are the subarcs of hypercircles which join two distinct points in \( H \). Finally, the geodesic triangles in \( H \) are triangles, formed in manner familiar, for which the three edges are geodesic arcs.
4° Now let $p$ and $q$ be any positive integers for which:

(*) \[ 4 < (p-2)(q-2) \]

Let $\alpha$, $\beta$, and $\gamma$ be the angular measures defined as follows:

\[ \alpha = \frac{\pi}{p}, \quad \beta = \frac{\pi}{q}, \quad \gamma = \frac{\pi}{2} \]

By (*), we find that:

\[ \alpha + \beta + \gamma < \pi \]

Let $T$ be the geodesic triangle in $H$ for which the measures of the vertex angles are $\alpha$, $\beta$, and $\gamma$. By the Laws of Cosines and Sines in hyperbolic geometry, the vertex angles of $T$ determine the edges. In the following figure, we display $T$ in standard position and we label the vertices of $T$ by $A$, $B$, and $C$, in correspondence with the measures $\alpha$, $\beta$, and $\gamma$ of the vertex angles.

![Figure 2: p=8, q=3](image)

5° Let us describe a method for constructing $T$. To that end, we introduce the angular measure $\delta$, defined as follows:

\[ \delta = \pi - (\alpha + \beta + \gamma) = \frac{\pi}{2} - (\alpha + \beta) \]
In turn, we produce the diagram:

![Diagram](image_url)

Figure 3

by the following steps. We draw the “horizontal” line passing through the points $A$ and $Z$, where $A$ is the center of $H$ and where $Z$ is a remote point in the exterior of $H$. We construct the point $F$ on the boundary of $H$ so that the measure of the angle $\angle ZAF$ is $\alpha$. We construct the point $H$ on the line segment $AZ$ so that the measure of the angle $\angle AFH$ is $\beta + \gamma$. Of course, the measure of the angle $\angle AHF$ is $\delta$. We draw the circle (in red), for which the center is $H$ and for which the line segment $HF$ is a radius. We draw the circle (in blue), for which the line segment $AH$ is a diameter. We obtain the points $E$, $G$, $I$, and $J$. Obviously, the measure of the angle $\angle AIH$ is $\gamma$. 

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At this point, the triangles $\triangle AFH$ and $\triangle AIH$ and the red and blue circles are the good effects of our work. Now we complete the construction by contracting these triangles and circles so that, in particular, the point $I$ coincides with the point $E$.

We can achieve this effect simply by constructing the point $D$ on the line segment $\overline{AH}$ so that the measure of the angle $\angle AED$ is $\gamma$. Proceeding mechanically, we draw the hypercircle (in red), for which the center is $D$ and for which the line segment $\overline{DE}$ is a radius. We draw the circle (in blue), for which the line segment $\overline{AD}$ is a diameter. Finally, we mark the points $B$ and $C$ of intersection of the hypercircle with the line segments $\overline{AF}$ and $\overline{AH}$, respectively. Clearly, the triangles $\triangle ABD$ and $\triangle AED$ are similar to the triangles...
\(\triangle AFH\) and \(\triangle AIH\), respectively. In this way, we obtain the geodesic triangle \(T\) for which the vertices are \(A\), \(B\), and \(C\) and for which the measures of the corresponding vertex angles are \(\alpha\), \(\beta\), and \(\gamma\), respectively.

7° We do not attempt to design a shortcut, by presuming, in error, that the critical point \(D\) lies on the line segment \(IJ\).

8° One can implement the foregoing construction of a geodesic triangle by straightedge and compass iff the vertex angles of the triangle are constructible. It is the same to say that the positive integers \(p\) and \(q\) are of the form:

\[
2^k \pi_1 \pi_2 \cdots \pi_\ell
\]

where \(k\) and \(\ell\) are nonnegative integers and where:

\[
\pi_1, \pi_2, \ldots, \pi_\ell
\]

are distinct Fermat primes. The latter are those which are prime among integers of the form:

\[
2^n + 1
\]

where \(n\) is a positive integer. Actually, such an integer is prime only if \(n\) is itself a power of 2. Currently:

\[
2^2 = 3, \ 2^2 = 5, \ 2^4 = 17, \ 2^8 = 257, \ 2^{11} = 65537
\]

are known to be Fermat primes, while:

\[
2^7, \ 2^9, \ 2^{10}, \ 2^{11}, \ 2^{12}
\]

are known to be composite.

9° The cases which figure in the Circle Limit Series of M. C. Escher are the following:

\[(p, q) = (6, 4) \quad \text{and} \quad (p, q) = (8, 3)\]