1° Let $X$ be a set, let $\mathcal{A}$ be a borel algebra of subsets of $X$, and let $\mu$ be a normalized measure defined on $\mathcal{A}$. One refers to the ordered triple:

$$(X, \mathcal{A}, \mu)$$

as a (normalized) measure space, but sometimes as a probability space. Let $T$ be a borel mapping carrying $X$ to itself for which $\mu$ is invariant:

$$T_\ast(\mu) = \mu$$

One refers to the ordered quadruple:

$$(X, \mathcal{A}, \mu, T)$$

as an (abstract) dynamical system.

2° Now let:

$$(X, \mathcal{A}, \mu)$$

be a probability space and let:

$$f_0, f_1, f_2, \ldots, f_j, \ldots$$

be a sequence of (real-valued) borel functions defined on $X$. One may just as well present the foregoing sequence as a (borel) mapping $F$ carrying $X$ to $\mathbb{R}^N$, defined as follows:

$$F(x) := (f_0(x), f_1(x), f_2(x), \ldots, f_j(x), \ldots) \quad (x \in X)$$

Obviously, one may recover the sequence:

$$f_0, f_1, f_2, \ldots, f_j, \ldots$$

from the mapping $F$ by applying the projections:

$$p_j(t) := t_j \quad (t = (t_0, t_1, t_2, \ldots, t_j, \ldots) \in \mathbb{R}^N)$$

carrying $\mathbb{R}^N$ to $\mathbb{R}$. Thus:

$$f_j = p_j \cdot F \quad (j \in \mathbb{N})$$
One refers to the ordered quadruple:

$$(X, \mathcal{A}, \mu, F)$$

as a random process. The various borel functions in the sequence:

$$f_0, f_1, f_2, \ldots, f_j, \ldots$$

are the random variables comprising the random process. For each nonnegative integer $j$, one defines the marginal distribution for $f_j$ as follows:

$$\nu_j := (f_j)_*(\mu)$$

Of course:

$$\nu_j$$

is a normalized measure on $\mathbb{R}$. One says that the random process is identically distributed (id) iff all the marginal distributions coincide:

$$\nu_j := \nu_0 \quad (j \in \mathbb{N})$$

For any finite strictly increasing sequence:

$$j_1 < j_2 < j_3 < \cdots < j_n$$

of nonnegative integers, one defines the joint marginal distribution as follows:

$$\nu_{j_1,j_2,\cdots,j_n} := (f_{j_1} \times f_{j_2} \times \cdots \times f_{j_n})_*(\mu)$$

Of course:

$$\nu_{j_1,j_2,\cdots,j_n}$$

is a normalized measure on (the borel algebra comprised of the borel subsets of) $\mathbb{R}^n$. One says that the random process is stationary iff, for any finite strictly increasing sequence:

$$j_1 < j_2 < j_3 < \cdots < j_n$$

of nonnegative integers and for any positive integer $k$:

$$\nu_{j_1,j_2,\cdots,j_n} = \nu_{k_1,k_2,\cdots,k_n}$$

where:

$$k_1 := j_1 + k, \ k_2 := j_2 + k, \ldots, k_n := j_n + k$$

In due course, we will reformulate this formidably abstract condition in more comprehensible “geometric” terms. Taking $n$ to be 1, one can readily check that if the random process is stationary then it is identically distributed.
One says that the random process is *independent* iff, for any finite strictly increasing sequence:

\[ j_1 < j_2 < j_3 < \ldots < j_n \]

of nonnegative integers:

\[ \nu_{j_1, j_2, \ldots, j_n} = \prod_{m=1}^{n} \nu_{j_m} \]

One can readily check that if the random process is independent and identically distributed (*iid*) then it is stationary. One sometimes refers to an iid random process as a *bernoulli* process.

3° Let:

\[(X, \mathcal{A}, \mu, T)\]

be a dynamical system and let:

\[ h \]

be a (real-valued) borel function defined on \( X \). One defines the corresponding random process:

\[(X, \mathcal{A}, \mu, F)\]

as follows:

\[(1)\]

\[ f_j := h \cdot T^j \]

Clearly:

\[ F(x) = (h(T^0(x)), h(T^1(x)), h(T^2(x)), \ldots, h(T^j(x)), \ldots) \quad (x \in X) \]

We may say that the ordered quintuple:

\[(X, \mathcal{A}, \mu, T, h)\]

comprised of the dynamical system:

\[(X, \mathcal{A}, \mu, T)\]

and the *observable*:

\[ h \]

defines the corresponding random process:

\[(X, \mathcal{A}, \mu, F)\]

by means of relation (1). One can readily show that this random process is stationary.
Conversely, let:

\[(X, \mathcal{A}, \mu, F)\]

be a random process. Let:

\[\nu\]

be the (normalized) measure defined on (the borel algebra \(\mathcal{B}\) comprised of the borel subsets of) \(\mathbb{R}^N\) as follows:

\[(2) \quad \nu := F_*(\mu)\]

Let \(\Sigma\) be the (borel) mapping carrying \(\mathbb{R}^N\) to itself, defined as follows:

\[(3) \quad \Sigma(t) := u = (u_0, u_1, u_2, \ldots, u_j, \ldots) \quad (t = (t_0, t_1, t_2, \ldots, t_j, \ldots) \in \mathbb{R}^N)\]

One can readily show that if the given random process is stationary then \(\nu\) is invariant for \(\Sigma\):

\[\Sigma_*(\nu) = \nu\]

In fact, the relation just stated provides a natural, rather more intuitive view of the condition that the given random process be stationary. We may say that the random process:

\[(X, \mathcal{A}, \mu, F)\]

if stationary, defines the dynamical system:

\[(\mathbb{R}^N, \mathcal{B}, \nu, \Sigma)\]

by means of relations (2) and (3). The observable:

\[p_0\]

completes the picture:

\[(\mathbb{R}^N, \mathcal{B}, \nu, \Sigma, p_0)\]

We may summarize the foregoing transitions in the following schematic form:

\[(X, \mathcal{A}, \mu, T, h) \rightarrow (X, \mathcal{A}, \mu, F) \rightarrow (\mathbb{R}^N, \mathcal{B}, \nu, \Sigma, p_0)\]

One should note that:

\[(X, \mathcal{A}, \mu, T, h)\]

and:

\[(\mathbb{R}^N, \mathcal{B}, \nu, \Sigma, p_0)\]
are closely related, in that the borel mapping $F$ carries $X$ to $\mathbb{R}^N$:

$$F: X \to \mathbb{R}^N$$

$F$ transforms $\mu$ to $\nu$:

$$F_*(\mu) = \nu$$

$F$ intertwines $T$ and $\Sigma$:

$$\Sigma \cdot F = F \cdot T$$

and $F$ transforms $p_0$ to $h$:

$$h = p_0 \cdot F$$

6° One may continue the process one more time. The dynamical system:

$$(\mathbb{R}^N, \mathcal{B}, \nu, \Sigma)$$

and the observable:

$$p_0$$

define the random process:

$$(\mathbb{R}^N, \mathcal{B}, \nu, I)$$

where $I$ is the identity mapping carrying $\mathbb{R}^N$ to itself. The corresponding sequence of random variables for this random process is the sequence of projections:

$$p_0, p_1, p_2, \ldots, p_j, \ldots$$

The relevant point is that:

$$p_j = p_0 \cdot \Sigma_j \quad (j \in \mathbb{N})$$

One should note that:

$$(X, \mathcal{A}, \mu, F)$$

and:

$$(\mathbb{R}^N, \mathcal{B}, \nu, I)$$

are closely related, in that the borel mapping $F$ carries $X$ to $\mathbb{R}^N$:

$$F: X \to \mathbb{R}^N$$

$F$ transforms $\mu$ to $\nu$:

$$F_*(\mu) = \nu$$

and $F$ transforms the sequence:

$$p_0, p_1, p_2, \ldots, p_j, \ldots$$

to the sequence:

$$f_0, f_1, f_2, \ldots, f_j, \ldots$$

which is to say that:

$$f_j = p_j \cdot F \quad (j \in \mathbb{N})$$