NONVANISHING OF DIRICHLET $L$-FUNCTIONS AT $s = 1$

In the proof of Dirichlet’s theorem on arithmetic progressions, after the various sums and products are unwound, and after what amounts to a simple piece of Fourier analysis, the crucial fact is that for any nontrivial Dirichlet character $\chi$,

$$L(\chi, s) \neq 0 \quad \text{at } s = 1.$$  

The fact can be proved in various ways. For example, our handout on Dirichlet’s theorem made use of cyclotomic arithmetic. Here we give, with some motivation, a more direct elementary argument, which admittedly is a bit ad hoc.

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#### 1. The argument when $\chi^2$ is nontrivial

For any $s \in \mathbb{C}$ such that $\Re(s) > 1$,

$$L(\chi, s) = \exp \log L(\chi, s) = \exp \log \prod_p (1 - \chi(p)p^{-s})^{-1}$$  

$$= \exp \sum_{p \in \mathcal{P}} \log(1 - \chi(p)p^{-s})^{-1} = \exp \sum_{p \in \mathcal{P}} \sum_{n \in \mathbb{Z}^+} \frac{\chi(p)^n}{np^{ns}}.$$  

Because in general $|\exp(z)| = \exp(\Re(z))$, it follows that for real $s > 1$,

$$|L(\chi, s)| = \exp \sum_{p, n} \frac{\cos(n\theta_p)}{np^{ns}} \text{ where } \chi(p) = e^{i\theta_p}.$$  

The cosines in the sum could well be positive or negative. However, modifying the calculation makes the summands nonnegative,

$$\zeta(s) L(\chi, s) = \exp \log(\prod_p (1 - p^{-s})^{-1}(1 - \chi(p)p^{-s})^{-1})$$  

$$= \exp \sum_{p \in \mathcal{P}} \log(1 - p^{-s})^{-1} + \log(1 - \chi(p)p^{-s})^{-1}$$  

$$= \exp \sum_{p \in \mathcal{P}} \sum_{n \in \mathbb{Z}^+} \frac{1 + \chi(p)^n}{np^{ns}}, \quad s > 1,$$

so that

$$|\zeta(s) L(\chi, s)| = \exp \sum_{p, n} \frac{1 + \cos(n\theta_p)}{np^{ns}}, \quad s > 1.$$
Now the summands are nonnegative, and thus
\[ |\zeta(s)L(\chi, s)| \geq 1, \quad s > 1. \]
This doesn’t give \( L(\chi, 1) \neq 0 \), though, because \( \zeta \) has a simple pole at \( s = 1 \), and so the previous display shows only that \( L(\chi, s) \) either is nonzero at \( s = 1 \) or has a simple zero at \( s = 1 \). Because a zero would force \( \zeta(s)L(\chi, s)^2 \) to vanish at \( s = 1 \), the next step is to study
\[ |\zeta(s)L(\chi, s)^2| = \exp \sum_{p,n} \frac{1 + 2\cos(n\theta_p)}{np^{ns}}, \quad s > 1, \]
but now we are back to a scenario where the terms of the sum need not be positive. To address this, the expression \( 1 + 2\cos(n\theta_p) \) can be augmented to a square by adding \( \cos^2(n\theta_p) \). Thus, consider
\[ L(\chi^2, s) = \exp \sum_{p,n} \frac{(\chi(p)^n)^2}{np^{ns}}, \]
so that, because \( \chi(p)^n = \cos(n\theta_p) + i\sin(n\theta_p) \), and hence \( (\chi(p)^n)^2 \) has real part \( \cos^2(n\theta_p) - \sin^2(n\theta_p) = 2\cos^2(n\theta_p) - 1 \),
\[ |L(\chi^2, s)| = \exp \sum_{p,n} \frac{2\cos^2(n\theta_p) - 1}{np^{ns}}, \quad s > 1. \]
Thus, more generally,
\[ |\zeta(s)^aL(\chi, s)^bL(\chi^2, s)^c| = \exp \sum_{p,n} \frac{a - c + b\cos(n\theta_p) + 2c\cos^2(n\theta_p)}{np^{ns}}, \quad s > 1. \]
Specialize to \( (a, b, c) = (3, 4, 1) \) to get
\[ |\zeta(s)^3L(\chi, s)^4L(\chi^2, s)| = \exp \sum_{p,n} \frac{2 + 4\cos(n\theta_p) + 2\cos^2(n\theta_p)}{np^{ns}} = \exp \sum_{p,n} \frac{2(1 + \cos(n\theta_p))^2}{np^{ns}} \geq 1, \quad s > 1, \]
so that
\[ \zeta(s)^3L(\chi, s)^4L(\chi^2, s) \text{ does not go to } 0 \text{ as } s \to 1^+. \]
But \( \zeta(s)^3 \) has a pole of order 3 at \( s = 1 \), and assuming that \( \chi^2 \) is not the trivial character, \( L(\chi^2, s) \) does not have a pole at \( s = 1 \). So the previous display shows that \( L(\chi, s) \) can’t have a zero at \( s = 1 \) if \( \chi^2 \) is nontrivial.

2. The argument when \( \chi^2 \) is trivial

The case where \( \chi^2 \) is trivial needs to be handled separately. Here we have
\[ \zeta(s)L(\chi, s) = \exp \sum_{p,n} \frac{1 + \chi(p)^n}{np^{ns}}, \quad \Re(s) > 1. \]
The sum in the previous display is a Dirichlet series \( D(s) \) with nonnegative coefficients,
\[ D(s) = \sum_{m \in \mathbb{Z}^+} \frac{a_m}{m^s}, \quad a_m = \begin{cases} (1 + \chi(p)^n)/n & \text{if } m = p^n, \\ 0 & \text{otherwise.} \end{cases} \]
Suppose that $L(\chi, 1) = 0$. Then consequently:

- The function $\zeta(s) L(\chi, s)$ is analytic on $\{\text{Re}(s) > 0\}$.
- The Dirichlet series $D(s)$ converges on $\text{Re}(s) > 1$ to a function $g(s)$ such that $\exp g(s) = \zeta(s) L(\chi, s)$. Landau’s lemma, below, says that consequently $\exp D(s) = \zeta(s) L(\chi, s)$ for $s \in (0, 1)$.
- However, when $n$ is even, $\chi(p)^n = 1$, and so for real $s > 1/2$,

$$D(s) \geq \sum_{p,n} 2 \frac{1}{np^{2s}} = \sum_{p,n} \frac{1}{np^{2s}} = \log(2s).$$

Thus $D(s) \to \infty$ as $s \to 1/2^+$. The third bullet contradicts the second, so the supposition $L(\chi, 1) = 0$ is untenable.

3. LANDAU’S LEMMA

**Proposition 3.1** (Weak version of Landau’s lemma). Suppose that a Dirichlet series with nonnegative coefficients,

$$D(s) = \sum_{n \geq 1} a_n n^{-s}, \quad a_n \geq 0 \text{ for all } n,$$

converges to an analytic function $f(s)$ on the open right half plane $\{\text{Re}(s) > \sigma_o\}$. Suppose that for some $\varepsilon > 0$, the function $f(s)$ extends analytically to the larger open right half plane $\{\text{Re}(s) > \sigma_o - \varepsilon\}$. Then the Dirichlet series $D(s)$ converges to $f(s)$ on the $x$-axis portion of the larger right half plane, i.e., $D(\sigma) = f(\sigma)$ for all $\sigma \in (\sigma_o - \varepsilon, \sigma_o)$.

**Proof.** By way of quick review, recall the basic definition

$$a^z = e^{z \log a}, \quad a \in \mathbb{R}^+, \quad z \in \mathbb{C},$$

so that the derivatives of $a^z$ are

$$(a^z)^{(k)} = (\log a)^k a^z, \quad a \in \mathbb{R}^+, \quad z \in \mathbb{C}, \quad k \in \mathbb{Z}_{\geq 0}.$$

Thus the power series expansion of $a^z$ about $z = 0$ is

$$a^z = \sum_{k \geq 0} \frac{(\log a)^k}{k!} z^k, \quad a \in \mathbb{R}^+, \quad z \in \mathbb{C},$$

Note that this is a small variant of the familiar series of $e^z$. We will refer back to this expansion later in the argument.

Returning to Landau’s lemma, we may translate the problem and take $\sigma_o = 0$. The translation leaves the Dirichlet series coefficients nonnegative. The function $f(s)$ is analytic on $B(1, 1 + \varepsilon)$. Thus for any $\sigma \in (-\varepsilon, 0)$ the power series representation of $f(s)$ about $s = 1$ converges at $\sigma$ to $f(\sigma)$,

$$f(\sigma) = \sum_{k \geq 0} \frac{f^{(k)}(1)}{k!} (\sigma - 1)^k = \sum_{k \geq 0} \frac{(-1)^k f^{(k)}(1)}{k!} (1 - \sigma)^k, \quad -\varepsilon < \sigma < 0.$$ 

Because the Dirichlet series $D(s) = \sum_{n \geq 1} a_n n^{-s}$ converges to $f(s)$ about $s = 1$, compute the summand-numerator $(-1)^k f^{(k)}(1)$ at the end of the previous display by differentiating $D(s)$ termwise,

$$(-1)^k f^{(k)}(1) = (-1)^k \sum_{n \geq 1} a_n (-\log n)^k n^{-s} \bigg|_{s=1} = \sum_{n \geq 1} a_n (\log n)^k \frac{1}{n}.$$
Thus the penultimate display is now

\[ f(\sigma) = \sum_{k \geq 0} \sum_{n \geq 1} \frac{a_n (\log n)^k}{k! n} (1 - \sigma)^k, \quad -\varepsilon < \sigma < 0. \]

All of the terms are nonnegative, so we may rearrange the sum,

\[ f(\sigma) = \sum_{n \geq 1} \frac{a_n}{n} \sum_{k \geq 0} \frac{(\log n)^k}{k!} (1 - \sigma)^k, \quad -\varepsilon < \sigma < 0. \]

As explained at the beginning of the proof, the inner sum is the power series expansion of \( n^s \) about 0 at \( s = 1 - \sigma \). Thus

\[ f(\sigma) = \sum_{n \geq 1} \frac{a_n}{n} n^{1 - \sigma} = \sum_{n \geq 1} a_n n^{-\sigma} = D(\sigma), \quad -\varepsilon < \sigma < 0. \]

This is the desired result. \( \square \)