1. Introduction

The topic of this lecture is \textit{eventually} the unique factorization theorem for the integers:

\textbf{Theorem 1.1.} Let $n$ be a nonzero integer. Then $n$ factors as

$$n = \pm p_1^{e_1} \cdots p_r^{e_r}, \quad r \geq 0, \ p_1, \cdots, p_r \in \mathbb{P}, \ e_1, \cdots, e_r \in \mathbb{Z}^+, \quad \text{and the factorization is unique.}$$

The proof that a factorization exists is easy, at least on the face of it. Consider any positive integer $n$. If $n$ is irreducible then we are done. Otherwise $n = n_1n_2$ with $n_1 < n$ and $n_2 < n$, and so we are done by induction. The only worrisome point here is that irreducible has appeared as a stand-in synonym for prime, suggesting that prime might mean something other than irreducible to the cognoscenti. We will see that indeed the two words mean different things, and that the mathematical use of prime is not as we would expect.

By contrast, the proof that the factorization is unique is nuanced. Many books prove unique factorization in $\mathbb{Z}$ by elementary methods, but to me the issues are somehow more naturally (i.e., more clearly, perhaps more easily) discussed in the context of ring theory rather than just in the integers $\mathbb{Z}$. The upshot is that this lecture in some sense proceeds backwards through chapter 1 of Ireland and Rosen. (Nonetheless, you should read the chapter from front to back.)

2. Rings, Integral Domains

\textbf{Definition 2.1.} A commutative ring with identity is an algebraic structure \((R, +, \cdot)\) that satisfies all of the field axioms except (possibly) the existence of multiplicative inverses. That is, for all $r, s, t \in R$ we have

$$r + s = s + r,$$
$$r \cdot s = s \cdot r,$$
$$r \cdot (s + t) = r \cdot s + r \cdot t.$$ 

and there exist distinct additive and multiplicative identities $0, 1 \in R$ such that for all $r \in R$,

$$r + 0 = r,$$
$$r + s = 0 \quad \text{for some } s \in R,$$
$$r \cdot 1 = r.$$
Often we will simply refer to a commutative ring with identity as a ring. And we usually omit the “·” symbol for multiplication. As in Math 112, for any given \( r \in R \), the element \( s \in R \) such that \( r + s = 0 \) is unique, and so it can be unambiguously denoted \(-r\).

**Definition 2.2.** Given a ring \((R, +, \cdot)\), its unit group is the algebraic structure \((R^\times, \cdot)\) whose underlying structure is the set of multiplicatively invertible elements of \( R \),
\[
R^\times = \{ r \in R : rs = 1 \text{ for some } s \in R \},
\]
and whose operation is the restriction of the multiplication of \( R \) to \( R^\times \).

The unit group is an abelian group in that the product of two units is a unit (if \( rs = 1 \) and \( r's' = 1 \) then \( (rr')(ss') = 1 \)) and that for all \( r, s, t \in R^\times \),
\[
rs = sr, \\
(rst) = (rs)t, \\
r1 = r,
\]
and for all \( r \in R^\times \),
\[
rs = 1 \text{ for some } s \in R^\times.
\]
The \( s \) here is unique, so it can be denoted \( r^{-1} \).

**Definition 2.3.** An integral domain is a ring \((R, +, \cdot)\) satisfying the following property:

\[
\text{For all } r, s \in R, \quad rs = 0 \implies r = 0 \text{ or } s = 0.
\]

That is, an integral domain has no zero-divisors, i.e., no nonzero elements \( r, s \) such that \( rs = 0 \). (To make sure that the language is clear: zero-divisor means divisor of zero.) The cancellation law holds in any integral domain:

\[
\text{If } ab = ac \text{ and } a \neq 0 \text{ then } b = c.
\]

Note that we do not prove the cancellation law by multiplying through by \( a^{-1} \)—the inverse may not exist. Rather, the argument is that \( a(b - c) = 0 \) and \( a \neq 0 \), so that \( b - c = 0 \).

Some rings to bear in mind, beyond the most obvious example \( \mathbb{Z} \) (we usually write \( R \) rather than \((R, +, \cdot)\) when the operations are clear) are

- the Gaussian integers \( \mathbb{Z}[i] \),
- the cubic integers \( \mathbb{Z}[\omega] \) where \( \omega = \zeta_3 = e^{2\pi i/3} = (-1 + i\sqrt{3})/2 \),
- the polynomial ring \( k[X] \) where \( k \) is any field.

Note that \( k[X] \) is a ring of functions rather than a ring of numbers.

3. Prime and Irreducible Elements

**Definition 3.1.** Let \( R \) be an integral domain. An element \( r \) of \( R \) divides an element \( s \) of \( R \) if \( s = rr' \) for some \( r' \in R \). The symbolic notation for \( r \) divides \( s \) is
\[
r \mid s.
\]

A nonunit \( r \) of \( R \) is prime if:

\[
\text{For all } s, s' \in R, \quad r \mid ss' \implies r \mid s \text{ or } r \mid s'.
\]
Let \( R^\times \) be the unit group of \( R \). A nonzero nonunit \( r \) of \( R \) is \textbf{irreducible} if:

For all \( s, s' \in R \), \( ss' = r \implies s \in R^\times \) or \( s' \in R^\times \).

Thus primality is a criterion about how a given ring element fits into products, while irreducibility is a criterion about how products fit into a given element. There are ways to rewrite the definitions of \textit{prime} and \textit{irreducible} to further emphasize their symmetry; for example, for irreducibility the condition

for all \( s, s' \in R \), \( ss' = r \implies r | s \) or \( r | s' \)

is equivalent to the condition in the definition, again excluding \( r = 0 \). (However, it deserves brief mention that in a general commutative ring with 1, as compared to our more specific environment of an integral domain, the two definitions just given for irreducibility are not equivalent. For example, in the ring \( \mathbb{Z}/6\mathbb{Z} \) the factorization \( 2 = 2 \cdot 4 \) shows that 2 does not satisfy the first definition, but 2 does satisfy the second.)

4. \textbf{Generally Prime }\implies\textbf{ Irreducible}

The cancellation law says that 0 is prime in any integral domain. On the other hand, 0 is not irreducible because the the definition of irreducibility excludes it. The next result says that 0 is exceptional in being a prime that is not irreducible.

**Proposition 4.1.** In any integral domain, nonzero primes are irreducible.

**Proof.** Let \( R \) be an integral domain and let \( R^\times \) be its unit group. Consider any nonzero prime \( r \) of \( R \). If \( r = ss' \) then certainly \( r | ss' \) and so without loss of generality \( r | s \). Thus \( r = ss' = rus' \). Since \( r \neq 0 \) we have \( us' = 1 \) by cancellation, and consequently \( s' \in R^\times \). \( \square \)

The converse question is

\textit{Are the irreducible elements of an integral domain prime?}

This question does not have a general answer. For example, consider a subring of the complex number system and its unit group,

\( R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}, \quad R^\times = \{\pm 1\}. \)

Then \( R \) is an integral domain. The element 2 of \( R \) is irreducible because for all \( s, s' \in R \),

\[ ss' = 2 \implies s\bar{s}s' = 4 \implies s\bar{s} = 2 \text{ (else } s \text{ or } s' \text{ is a unit),} \]

but the condition \( s\bar{s} = 2 \) is impossible in \( R \) since \( s\bar{s} = a^2 + 5b^2 \) for some \( a, b \in \mathbb{Z} \). On the other hand 2 is not prime because

\[ 2 | 6 = (1 + \sqrt{-5})(1 - \sqrt{-5}) \text{ but } 2 \nmid (1 + \sqrt{-5}) \text{ and } 2 \nmid (1 - \sqrt{-5}). \]

(For instance, if \( 2 | 1 + \sqrt{-5} \) then also \( 2 = \frac{2}{1 + \sqrt{-5}} = 1 - \sqrt{-5} \), and consequently \( 4 = 2 \cdot 2 | (1 + \sqrt{-5})(1 - \sqrt{-5}) = 6 \), which is false.)
5. Euclidean Domains

**Definition 5.1.** The integral domain $R$ is Euclidean if it comes equipped with a norm function

$$N: R \setminus \{0\} \to \mathbb{Z}_{\geq 0}$$

such that the following condition holds: For all $a, b \in R$ with $b \neq 0$, there exist $q, r \in R$ such that

$$a = qb + r, \quad r = 0 \text{ or } Nr < Nb.$$ 

Here $q$ is the quotient obtained on dividing $a$ by $b$, and $r$ is the remainder.

All of our example rings from earlier are Euclidean.

- For $R = \mathbb{Z}$, take $Nn = |n|$ for all nonzero $n \in \mathbb{Z}$.
- For $R = \mathbb{Z}[i]$, take $Nz = z\bar{z} = |z|^2 = a^2 + b^2$ for all nonzero $z = a + ib \in \mathbb{Z}[i]$.
- For $R = \mathbb{Z}[\omega]$, take $Nz = z\bar{z} = |z|^2 = a^2 - ab + b^2$ for all nonzero $z = a + \omega b \in \mathbb{Z}[\omega]$. (Note that $\omega = \omega^2$.)
- For $R = k[X]$, take $Nf = \deg(f)$ for nonzero polynomials $f \in k[X]$. Note that the nonzero constant polynomials have norm 0. This norm is additive rather than multiplicative. Instead we could take $Nf = 2^{\deg(f)}$ to get a multiplicative norm with the nonzero constant polynomials having norm 1, and extend to $N0 = 0$ (apparently the norm of 0 was somehow $-\infty$ a moment ago), or we could take $Nf = a^{\deg(f)}$ for any integer $a > 1$ instead of $a = 2$.

(Warning: Ireland and Rosen are a little casual on pages 12–13. They extend the norms here to $\mathbb{Q}[i]$ and $\mathbb{Q}[\omega]$ in sections 1.4.1 and 1.4.2. In 1.4.1 they have $\lambda \in \mathbb{Q}[i]$ but then $\alpha, \gamma \in \mathbb{Z}[i]$. In 1.4.2 they define $\lambda \in \mathbb{Z}[\omega]$ but then use it in $\mathbb{Q}[\omega]$.)

In each case we need to verify that the specified norm makes the integral domain Euclidean.

- Verifying that the $\mathbb{Z}$-norm makes $\mathbb{Z}$ Euclidean is easy: Given $a, b \in \mathbb{Z}$ with $b \neq 0$, let

$$S = \{a - qb : q \in \mathbb{Z}\}.$$ 

Note that $S$ contains nonnegative elements (those arising from $q \leq a/b$ if $b > 0$, those arising from $q \geq a/b$ if $b < 0$), and let $r$ be the least nonnegative element of $S$. Then indeed $a = qb + r$, and $Nr = r$ must be less than $b$ because otherwise $S$ has a smaller nonnegative element $r - b$.

- To verify that the $\mathbb{Z}[i]$-norm makes $\mathbb{Z}[i]$ Euclidean, consider any $a, b \in \mathbb{Z}[i]$ with $b \neq 0$. Note that $a/b = r + is$ where $r, s \in \mathbb{Q}$. (Here we are tacitly using the fact that $\mathbb{Z}(i) = \mathbb{Q}(i) = \mathbb{Q}[i]$.) Then $a/b = r + is$ sits in the unit box about some point $q = m + in \in \mathbb{Z}[i]$. Consequently $a/b - q$ sits in the unit box about 0. It follows that $N(a/b - q) \leq 1/2 < 1$, and so since the norm (extended to $\mathbb{Q}[i]$) is multiplicative, $N(a - qb) < Nb$. Alternatively, to work only in $\mathbb{Z}[i]$, one can argue that given $a, b \in \mathbb{Z}[i]$ with $b \neq 0$, some translate $r = a - qb$ lies in the square having side-midpoints $\pm b/2$ and $\pm ib/2$. The points of the square having maximal norm are the corners, of norm $Nb/2$.

- The verification that $\mathbb{Z}[\omega]$ is Euclidean is very similar. This time we have a hexagon about 0, and any point in the hexagon has norm at most 1/3, hence strictly less than 1.
The verification that the \( k[\mathcal{X}] \)-norm makes \( k[\mathcal{X}] \) Euclidean is a matter of polynomial long division. Specifically, given \( f, g \in k[\mathcal{X}] \) with \( g \neq 0 \), proceed as follows.

- \((\text{Initialize})\)
  - Set \( q = 0 \) and \( r = f \). (So \( f = qg + r \).
  - Let \( g = b_m x^m + \cdots \).
- \((\text{Iterate})\)
  - While \( \deg r \geq \deg g \),
    - let \( r = r_n x^n + \cdots \) and set \( \delta = (r_n/b_m)x^n-m \)
    - replace \( q \) by \( q + \delta \)
    - replace \( r \) by \( r - \delta g \). (Still \( f = qg + r \), and \( \deg r \) has decreased.)
- \((\text{Terminate})\)
  - Return \( q \) and \( r \). (Now \( f = qg + r \), and \( \deg r < \deg g \).)

Similarly to \( \mathbb{Z}[i] \) one can show that \( \mathbb{Z}[\sqrt{-2}] \) is Euclidean, and similarly to \( \mathbb{Z}[\omega] \) one can show that \( \mathbb{Z}[(-1+i\sqrt{7})/2] \) and \( \mathbb{Z}[(-1+i\sqrt{11})/2] \) are Euclidean.

6. Ideals, Principal Ideals

**Definition 6.1.** Let \( R \) be a ring. A subset \( I \) of \( R \) is called an **ideal** if

1. \( I \) is closed under addition: for all \( i, j \in I \), also \( i + j \in I \).
2. \( I \) is strongly closed under multiplication: for all \( r \in R \) and \( i \in I \), also \( ri \in I \).

The definition of ideal may seem unmotivated. One explanation is that a ring is sort of like a vector space over itself in that pairs of elements can be added and any element can be multiplied by the ring elements; from this point of view an ideal is analogous to a subspace. A second explanation is that ideals are the correct subrings for the creation of quotient rings, just as normal subgroups are the correct subgroups for the creation of quotient groups. But we do not discuss quotient structures here.

For example, let \( R \) be a ring, pick any element \( r_0 \in R \) and let \( (r_0) \) denote the set of all multiples of \( r_0 \),

\[
(r_0) = \{rr_0 : r \in R\}.
\]

For instance, \( (2) = \{0, \pm 2, \pm 4, \cdots \} \) in \( \mathbb{Z} \). Similarly, pick any two elements \( r_0, s_0 \in R \) and let \( (r_0, s_0) \) denote the set of all \( R \)-linear combinations of \( r_0 \) and \( s_0 \),

\[
(r_0, s_0) = \{rr_0 + ss_0 : r, s \in R\}.
\]

For instance, \( (2, 3) = \{-1 \cdot 2 + 1 \cdot 3, \cdots \} = \{1, \cdots \} = \mathbb{Z} \) in \( \mathbb{Z} \), while \( (2, 4) = (2) \) and \( (4, 6) = (2) \) as well.

**Definition 6.2.** The ideal \( I \) is **principal** if it takes the form \( I = (r) \) for some \( r \in R \).

So far, all of the ideals that we have seen are principal. (Please do not write “principle ideal.”) For an example of a nonprincipal ideal, let \( R = \mathbb{Z}[\sqrt{5}] \) and let

\[
I = (2, 1 + \sqrt{5}).
\]

Recalling that \( N(a + b\sqrt{5}) = a^2 + 5b^2 \), note that \( N2 = 4 \) and \( N(1 + \sqrt{5}) = 6 \).

Suppose that \( I \) is principal, i.e., \( I = (r) \) for some \( r \in \mathbb{Z}[\sqrt{-5}] \). Then

\[
2 = rs \text{ for some } s, \text{ so } Nr \mid N2 = 4.
\]
and

\[ 1 + \sqrt{-5} = rs' \text{ for some } s', \text{ so } Nr \mid N(1 + \sqrt{-5}) = 6. \]

Thus \( Nr \mid 2 \). But the condition \( Nr = 2 \) is impossible. And the condition \( Nr = 1 \) forces \( I = \mathbb{Z}[\sqrt{-5}] \), which is false. (Any element of \( I \) is \( r = 2s + (1 + \sqrt{-5})t \) where \( s, t \in \mathbb{Z}[\sqrt{-5}] \), and a little algebra shows that \( r = a + b\sqrt{-5} \) where \( a \) and \( b \) have the same parity. Thus \( r \neq 1 \), i.e., \( 1 \notin I \), and so the ideal is not the full ring.) In conclusion, \( I \) can not be a principal ideal.

**Definition 6.3.** An integral domain in which every ideal is principal is called a principal ideal domain. Principal ideal domain is usually abbreviated to PID.

7. Euclidean \( \Rightarrow \) PID

**Proposition 7.1.** Every Euclidean domain is a PID.

**Proof.** Let \( R \) be a Euclidean domain. Let \( I \) be a nonzero ideal in \( R \). Let \( b \in I \) be an element of least norm. Note that \( b \neq 0 \). Since \( R \) is Euclidean, we have for any \( a \in I \) some \( q, r \in R \) such that

\[ a = qb + r, \quad r = 0 \text{ or } Nr < Nb. \]

Since \( I \) is an ideal and \( a, b \in I \), also \( r \in I \), making the condition \( Nr < Nb \) impossible. Thus \( r = 0 \), and so \( a = qb \). That is, every element of \( I \) is a multiple of \( b \) and hence \( I = (b) \). \( \square \)

As an application of the proposition, consider a PID \( R \). For any \( x, y \in R \) the ideal \( (x,y) \) takes the form \( (x,y) = (z) \) for some \( z \in R \). The ideal-generator \( z \) is a common divisor of \( x \) and \( y \). Also, \( z \) is a linear combination of \( x \) and \( y \),

\[ z = ax + by \quad \text{for some } a, b \in R. \]

The display shows that any common divisor \( w \) of \( x \) and \( y \) divides \( z \). Thus \( z \) is the greatest common divisor of \( x \) and \( y \). That is, the greatest common divisor of \( x \) and \( y \) is a linear combination of \( x \) and \( y \) that generates the ideal \( (x,y) \). In symbols, \( (x,y) = (\gcd(x,y)) \). For this reason, the greatest common divisor \( \gcd(x,y) \) is usually written \( (x,y) \) without the \( \gcd \); the collision of \( (x,y) \) as \( \gcd \)-notation and as ideal-notation is a nonissue because the \( \gcd \) and the ideal are essentially the same thing. We saw this earlier with our calculations that \( (2,3) = (1) \), \( (2,4) = (2) \), and \( (4,6) = (2) \) in \( \mathbb{Z} \).

Continuing with the ideas of the previous paragraph, we compute a \( \gcd \) by finding an ideal-generator,

\[
\begin{align*}
(826, 1890) &= (826, 1890 - 2 \cdot 826) \\
&= (238, 826) = (238, 826 - 3 \cdot 238) \\
&= (112, 238) = (112, 238 - 2 \cdot 112) \\
&= (14, 112) = (14, 112 - 8 \cdot 14) \\
&= (0, 14) = (14).
\end{align*}
\]

Thus \( \gcd(826, 1890) = 14 \). (The process just demonstrated is the venerable Euclidean algorithm). And furthermore, we can backtrack to express the \( \gcd \) as a
linear combination of the two given numbers,

\[ 14 = 238 - 2 \cdot 112 = 238 - 2 \cdot (826 - 3 \cdot 238) = 7 \cdot 238 - 2 \cdot 826 = 7 \cdot (1890 - 2 \cdot 826) - 2 \cdot 826 = -16 \cdot 826 + 7 \cdot 1890. \]

Or, alternatively to backtracking this way we can maintain a little more information through the forward Euclidean algorithm in order to have the linear combination coefficients at hand as well when it terminates. The method to do so is laid out in the first problem set for this course. The Euclidean algorithm shows that we know how to solve any equation of the form

\[ ax + by = c, \]

where \( a, b, c \in \mathbb{Z} \) are the given coefficients and we seek integer solutions \( (x, y) \). Solutions exist if and only if \( \gcd(a, b) \mid c \), in which case we can find one particular solution via the algorithm. All other solutions differ from the particular solution by solutions to the homogenized equation \( ax + by = 0 \), which is easy to solve: after dividing \( a \) and \( b \) by their \( \gcd \) we get \( a'x + b'y = 0 \) where \( \gcd(a', b') = 1 \), and so the solutions are \( (x, y) = n(b', -a') \) for all \( n \in \mathbb{Z} \).

8. PID \( \implies \) (Irreducible \( \implies \) Prime)

**Proposition 8.1.** Let \( R \) be a PID. Then every irreducible element of \( R \) is prime.

**Proof.** Let \( r \in R \) be irreducible. Suppose that \( r \mid ss' \) and \( r \nmid s \). We need to show that \( r \mid s' \). Since \( r \) is irreducible and \( r \nmid s \), in fact \( (r, s) = (1) \). Thus there exist \( a, b \in R \) such that \( ar + bs = 1 \). Consequently \( ars' + bss' = s' \). But \( r \mid ars' + bss' \), and hence \( r \mid s' \) as desired. \( \square \)

9. PID \( \implies \) Noetherian

**Definition 9.1.** A ring \( R \) is **Noetherian** if any ascending chain of ideals in \( R \),

\[ I_1 \subset I_2 \subset I_3 \subset \cdots, \]

eventually stabilizes, meaning that the \( I_n \) are equal for all \( n \) after some starting index \( N \).

**Proposition 9.2.** Let \( R \) be a PID. Then \( R \) is Noetherian.

**Proof.** Given an ascending chain of ideals in \( R \),

\[ I_1 \subset I_2 \subset I_3 \subset \cdots, \]

let

\[ I = \bigcup_{n=1}^{\infty} I_n. \]

Then \( I \) is an ideal of \( R \) (exercise). Since \( R \) is a PID, in fact \( I = (r) \) for some \( r \in R \). Since \( r \in I \), in fact \( r \in I_N \) for some \( N \). Thus \( I = (r) \subset I_N \subset I \), so that \( I_N = I \). Consequently \( I_n = I \) for all \( n \geq N \). \( \square \)
10. (Noetherian and (Irreducible $\Rightarrow$ Prime)) $\Rightarrow$ UFD

**Definition 10.1.** An integral domain in which every nonzero element factors as a product of finitely many irreducible elements, with the irreducible elements determined uniquely up to units, is called a unique factorization domain, or UFD.

**Proposition 10.2.** Let $R$ be a Noetherian integral domain in which all irreducible elements are prime. Then $R$ is a UFD.

*Proof.* Let $r$ be a nonzero element of $R$. The Noetherian property of $R$ gives a factorization of $r$ into finitely many irreducibles, since otherwise we could create a nonstabilizing chain of ideals,

$$ (r) \subset (r_1) \subset (r_2) \subset (r_3) \subset \ldots. $$

The fact that irreducibles are prime makes the factorization unique, since if

$$ r = up_1^{e_1} \cdots p_s^{e_s} = vq_1^{f_1} \cdots q_t^{f_t}, $$

where $u, v$ are units, and all $p_i$ and $q_j$ are irreducible, then

$$ p_1 | q_1^{f_1} \cdots q_t^{f_t}, $$

so that since $p_1$ is prime we have (after reindexing if necessary)

$$ p_1 | q_1 $$

and thus, after multiplying $q_1$ by a unit if necessary, $p_1 = q_1$. Repeat the argument from here starting from

$$ r/p_1 = up_1^{e_1} \cdots p_s^{e_s} = vq_1^{f_1} \cdots q_t^{f_t}. $$

By induction on $\sum e_i$ we have $s \leq t$ and $e_i \leq f_i$ for $i = 1, \ldots, s$. But by symmetry we also have $t \leq s$ and $f_i \leq e_i$ for $i = 1, \ldots, t$. Thus the factorization is unique. \( \square \)

11. Summary

We have shown that for integral domains,

| Euclidean $\Rightarrow$ PID $\Rightarrow$ \{ irreducible $\Rightarrow$ prime \} $\Rightarrow$ Noetherian $\Rightarrow$ UFD. |

And our integral domains $\mathbb{Z}$, $\mathbb{Z}[i]$, $\mathbb{Z}[\omega]$, and $k[X]$ are all Euclidean, so they are UFDs.

This all may seem pointlessly Byzantine, but the issues here are already live in elementary contexts. For example,

- Math 112 exercises have to dance around the question

  *For what positive integers $n$ is $\mathbb{Z}/n\mathbb{Z}$ a field?*

  Everybody knows morally that the answer is *For prime $n$. However, the problem is that while this isn’t hard to show for our revised notion of prime, showing that the answer is what we would phrase *For irreducible $n* raises the issue of whether prime and irreducible mean the same thing. For any $n \in \mathbb{Z}^+$ it is easy to show that

  $$ n \text{ is prime } \iff \mathbb{Z}/n\mathbb{Z} \text{ is a field } \iff n \text{ is irreducible}. $$

  But to show that if $n$ is irreducible then $\mathbb{Z}/n\mathbb{Z}$ is a field requires the non-trivial fact that irreducibles are prime in $\mathbb{Z}$. 

Similarly, any argument that there is no square root of 2 in \( \mathbb{Q} \) tacitly makes use of unique factorization. Ultimately the argument boils down to

Let \( r \in \mathbb{Q} \) satisfy \( r^2 = 2 \). Then \( r = 2^e r' \) where \( e \in \mathbb{Z} \) and \( r' \in \mathbb{Q} \) has no 2's in its numerator or its denominator. Thus \( 2 = r^2 = 2^{2e} (r')^2 \). But this is impossible because there are no 2's in \((r')^2\) and so the left side has one power of 2 while the right side has an even number of 2's.

The problem with the argument is that without unique factorization, a number with one power of 2 conceivably could equal a number with an even number of 2's.