Mathematics 361: Number Theory
Assignment #2

Reading: Ireland and Rosen, Chapter 2 (including the exercises)

Problems:

Even perfect numbers (nobody knows if there are any odd ones):
1. (a) Show that if $2^p - 1$ is prime (forcing $p$ to be prime) then $2^{p-1}(2^p - 1)$ is perfect.
   
   (b) If $m$ is even and perfect, show that $m$ takes the form $m = 2^{p-1}(2^p - 1)$ where $2^p - 1$ is prime. (Write $m = 2^{p-1}t$ where $p \geq 2$ and $t$ is odd and we don’t yet know whether $p$ is prime. Show that $\sigma(t) = 2^pr$ where $r$ is odd and $t = (2^p - 1)r$. Note that $r$ and $t$ are distinct factors of $t$. Use this to show that $r = 1$ and $2^p - 1$ is prime.)

   (c) Where does the argument in (b) break down if $p = 1$? That is, why can’t we argue as in (b) to show that there are no odd perfect numbers?

2. Work Ireland and Rosen, Exercises 1.30, some of 1.32—1.38.

More ring theory:
3. Let $R$ be a commutative ring. Consider an increasing chain of ideals in $R$,
   $$I_0 \subset I_1 \subset I_2 \subset \cdots .$$
Let $I = \bigcup_{n \geq 0} I_n$. Show that $I$ is an ideal in $R$.

4. In class we showed that if $R$ is an integral domain in which every irreducible element is prime and for which the Noetherian property holds, then $R$ is a UFD. Conversely, show that if $R$ is a UFD then every irreducible element is prime and the Noetherian property holds for principal ideals. (Warning: there exist UFDs for which the Noetherian property fails.)