MATHEMATICS 332: ALGEBRA — ASSIGNMENT 2

Reading: Gallian, chapters 0, 1, 2, 3, 10

Problems:

1. Let $n$ be a positive integer. Is each of the following subsets of $\text{GL}_n(\mathbb{R})$ a subgroup? When the answer is no, give some explanation. For one part, the answer depends on $n$.
   (a) The set of symmetric matrices $(a_{ji} = a_{ij})$ in $\text{GL}_n(\mathbb{R})$.
   (b) The set of trace-zero matrices $(\sum_{i=1}^n a_{ii} = 0)$ in $\text{GL}_n(\mathbb{R})$.
   (c) The set of upper-triangular matrices $(a_{ij} = 0$ if $i > j)$ in $\text{GL}_n(\mathbb{R})$.
   (d) The set of diagonal matrices $(a_{ij} = 0$ if $i \neq j)$ in $\text{GL}_n(\mathbb{R})$.

2. Let $c \in \mathbb{R}_{>0}$ be a positive real number. Consider the set-with-operation $(G_c, \cdot)$ where the set is
   
   $$G_c = \mathbb{Z} \times \mathbb{R} = \{(m, x) : m \in \mathbb{Z}, x \in \mathbb{R}\},$$

   and the operation is (omitting the “.”)
   
   $$(m, x)(n, y) = (m + n, x + cy).$$

   (a) Show that $(G_c, \cdot)$ is a group. (Don’t bother explaining why the operation returns values in the same set.)
   (b) For which (if any) values of $c$ is the group commutative?
   (c) For which (if any) values of $c$ does the nonempty subset $\mathbb{Z} \times \mathbb{Q}$ of $G_c$ form a subgroup under the operation?

3. Let $G$ be a group such that $(a \cdot b)^2 = a^2 \cdot b^2$ for all $a, b \in G$. Show that $G$ is commutative.

4. For any pair of real numbers $x$ and $y$, define the double embedding of the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$ as another column vector having three entries,

   $$\iota : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad \iota \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}.$$ 

   (a) Does $\iota$ surject? Does $\iota$ map to a basis of $\mathbb{R}^3$?
   (b) Show that there exists a unique function

   $$f : \text{GL}_2(\mathbb{R}) \rightarrow \text{GL}_3(\mathbb{R})$$

   such that the following diagram commutes:

   $\begin{array}{ccc}
   \text{GL}_2(\mathbb{R}) \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\
   (f, \iota) \downarrow & & \downarrow \iota \\
   \text{GL}_3(\mathbb{R}) \times \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3
   \end{array}$
That is, the desired relation is \( f(m) \cdot (\iota v) = \iota (m \cdot v) \) for all \( m \in \text{GL}_2(\mathbb{R}) \) and \( v \in \mathbb{R}^2 \).

(If your solution does not cite part (a) then it can not possibly be complete. Also, don’t forget to show that \( f(m) \) lies in \( \text{GL}_3(\mathbb{R}) \), i.e., that it is invertible.)

(c) Show that \( f \) is a homomorphism from \( \text{GL}_2(\mathbb{R}) \) to \( \text{GL}_2(\mathbb{R}) \). (Again, if your solution does not cite part (a) then it can not possibly be complete.)

5. (a) Let \( n \) be a positive integer. Let \( h \in \text{GL}_n(\mathbb{C}) \) be hermitian, meaning that \( h^* = h \) where \( h^* \) is the transpose-conjugate of \( h \). Consider the set of matrices that preserve the inner product defined by \( h \),

\[
U(h) = \{ m \in \text{GL}_n(\mathbb{C}) : m^* hm = h \}.
\]

Show that \( U(h) \) is a subgroup of \( \text{GL}_n(\mathbb{C}) \). This subgroup is the **unitary group** of \( h \).

(b) Show that the map

\[
f : \text{GL}_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}), \quad m \mapsto m^{-\top}
\]

is an endomorphism of \( \text{GL}_n(\mathbb{C}) \), and in fact an automorphism because it is its own inverse. (Such an automorphism is called an **involution**.)

(c) Show that the restriction of the map \( f \) from part (b) to \( U(h) \) gives an isomorphism

\[
f : U(h) \xrightarrow{\sim} U(h^{-1}).
\]