Reading and Exercises:

The special linear group over the integers is
\[ \text{SL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\}. \]

The complex upper half plane is
\[ \mathcal{H} = \{ \tau \in \mathbb{C} : \text{Im} (\tau) > 0 \}. \]

The group \( \text{SL}_2(\mathbb{Z}) \) acts on the set \( \mathcal{H} \) by fractional linear transformations,
\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \tau \mapsto a\tau + b \quad \frac{c\tau + d}{c\tau + d}. \]

Suppose that a point \( \tau \in \mathcal{H} \) is an elliptic point, meaning that it is fixed by a nontrivial transformation \( \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}). \) Thus
\[ a\tau + b = c\tau^2 + d\tau; \]
solving for \( \tau \) with the quadratic equation \( (c = 0 \text{ is impossible since } \tau \notin \mathbb{Q}) \) and remembering that \( \tau \in \mathcal{H} \) shows that
\[ |a + d| < 2 \text{ (exercise)}. \]

Thus the characteristic polynomial of \( \gamma \) is \( x^2 + 1 \) or \( x^2 \pm x + 1 \). Since \( \gamma \) satisfies its characteristic polynomial, one of \( \gamma^4 = I, \gamma^3 = I, \gamma^6 = I \) holds, and \( \gamma \) has order 1, 2, 3, 4, or 6 as a matrix. Orders 1 and 2 give the identity transformation (exercise). So the following proposition describes all nontrivial fixing transformations.

**Proposition.** Let \( \gamma \in \text{SL}_2(\mathbb{Z}). \)

(a) If \( \gamma \) has order 3 then \( \gamma \) is conjugate to \( \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \pm 1 \) in \( \text{SL}_2(\mathbb{Z}). \)

(b) If \( \gamma \) has order 4 then \( \gamma \) is conjugate to \( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \pm 1 \) in \( \text{SL}_2(\mathbb{Z}). \)

(c) If \( \gamma \) has order 6 then \( \gamma \) is conjugate to \( \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \pm 1 \) in \( \text{SL}_2(\mathbb{Z}). \)

**Proof.** (c) Since \( \gamma^6 = I \) the lattice \( L = \mathbb{Z}^2 \) of integral column vectors is a module over the ring \( \mathbb{Z}[\zeta_6] \) where \( \zeta_6 = e^{2\pi i/6} \): the scalar-by-vector product \( (a + b\zeta_6) \cdot v \) for \( a, b \in \mathbb{Z} \) and \( v \in L \) is defined to be the matrix-by-vector product \( (aI + b\gamma)v \). (In fact we are viewing \( L \) as a 2-dimensional representation of the cyclic group of order 6 generated by \( \zeta_6 \), the action of the group on \( L \) extended to the group algebra \( \mathbb{Z}[\zeta_6]. \))

The ring \( \mathbb{Z}[\zeta_6] \) is known to be a principal ideal domain and \( L \) is finitely generated over it. The structure theorem for finitely generated modules over a principal ideal domain therefore says that \( L \) is \( \mathbb{Z}[\zeta_6] \)-isomorphic to a sum \( \bigoplus_k \mathbb{Z}[\zeta_6]/I_k \) where the \( I_k \) are ideals. As an abelian group, \( L \) is free of rank 2. Every nonzero ideal \( I_k \) of \( \mathbb{Z}[\zeta_6] \) has rank 2 as an abelian group, making the quotient \( \mathbb{Z}[\zeta_6]/I_k \) a torsion group, so no such terms appear in the sum. Only one free summand appears, for otherwise the sum would be too big as an abelian group. Thus there is a \( \mathbb{Z}[\zeta_6] \)-module isomorphism \( \phi_\gamma : \mathbb{Z}[\zeta_6] \longrightarrow L. \)

Let \( u = \phi_\gamma(1) \) and \( v = \phi_\gamma(\zeta_6) \) and let \( [u \ v] \) denote the matrix with columns \( u \) and \( v \). Then \( L = Zu + Zv \) so \( \det[u \ v] \in \{ \pm 1 \} \) (exercise). Compute that \( \gamma u = \zeta_6 \cdot \)
\( \phi_\gamma(1) = \phi_\gamma(\zeta_6) = v, \) and similarly \( \gamma v = \zeta_6 \cdot \phi_\gamma(\zeta_6) = \phi_\gamma(-1+\zeta_6) = -u+v. \) Thus
\[
\gamma[u \ v] = [v \ -u+v] = [u \ v] \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \text{ so } \gamma = [u \ v] \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}^{-1},
\]
and
\[
\gamma[v \ u] = [-u+v \ v] = [v \ u] \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \text{ so } \gamma = [v \ u] \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}^{-1} [v \ u]^{-1}.
\]
One of \([u \ v], [v \ u]\) is in \( \text{SL}_2(\mathbb{Z}) \), proving (c). Parts (a) and (b) are similar (exercise).

Now we can understand elliptic points and their isotropy (stabilizing) subgroups.

**Corollary.** The elliptic points for \( \text{SL}_2(\mathbb{Z}) \) are \( \text{SL}_2(\mathbb{Z})i \) and \( \text{SL}_2(\mathbb{Z})\zeta_3 \) where \( \zeta_3 = e^{2\pi i/3} \). The modular curve \( Y(1) = \text{SL}_2(\mathbb{Z})\backslash \mathcal{H} \) has two elliptic points. The isotropy subgroups of \( i \) and \( \zeta_3 \) are
\[
\text{SL}_2(\mathbb{Z})i = \langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \rangle \quad \text{and} \quad \text{SL}_2(\mathbb{Z})\zeta_3 = \langle \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \rangle.
\]
For each elliptic point \( \tau \) of \( \text{SL}_2(\mathbb{Z}) \) the isotropy subgroup \( \text{SL}_2(\mathbb{Z})\tau \) is finite cyclic.

**Proof.** The fixed points in \( \mathcal{H} \) of the matrices in the proposition are \( i \) and \( \zeta_3 \). The first statement follows (exercise). The second statement follows since \( i \) and \( \zeta_3 \) are not equivalent under \( \text{SL}_2(\mathbb{Z}) \). The third statement can be verified directly (exercise), and the fourth statement follows since all other isotropy subgroups of order greater than 2 are conjugates of \( \text{SL}_2(\mathbb{Z})i \) and \( \text{SL}_2(\mathbb{Z})\zeta_3 \). See the guided exercise below for a more conceptual proof of the fourth statement.

**Exercise 1.** If the nontrivial transformation \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) \) fixes \( \tau \in \mathcal{H} \), show that \( |a+d| < 2 \).

**Exercise 2.** Show that if \( \gamma \in \text{SL}_2(\mathbb{Z}) \) has order 2 then \( \gamma = -I \). (One way to do this by thinking about the minimal polynomial of \( \gamma \), the polynomial \( X^2 - 1 \), and the characteristic polynomial of \( \gamma \); another way is to use the Jordan form of \( \gamma \).)

**Exercise 3.** (a) In the proof of the proposition, why does the condition \( L = Zu+Zv \) imply \( \det[u \ v] = \pm 1? \)
(b) Prove the other two parts of the proposition. (Hints are available from the instructor.)

**Exercise 4.** (a) Complete the proof of the corollary. (Hints are available from the instructor.)
(b) Give a more conceptual proof of the fourth statement of the corollary as follows: The isotropy subgroup of \( i \) in \( \text{SL}_2(\mathbb{R}) \) is the special orthogonal group \( \text{SO}(2) \), and it follows (you may take this as given for now) that \( \mathcal{H} \cong \text{SL}_2(\mathbb{R})/\text{SO}(2) \). Here the right side is a quotient space, not a quotient group. Show that an element \( s(\tau) \) of \( \text{SL}_2(\mathbb{R}) \) moves \( \tau \) to \( i \) and the isotropy subgroup of \( \tau \) correspondingly conjugates to a discrete subgroup of \( \text{SO}(2) \). But since \( \text{SO}(2) \) are the rotations of the circle, any such subgroup is cyclic.