COSETS IN LAGRANGE'S THEOREM AND IN GROUP ACTIONS

1. Lagrange’s Theorem

Let \( G \) be a group and \( H \) a subgroup, not necessarily normal.

**Definition 1.1 (Left \( H \)-equivalence).** Two group elements \( g, g' \in G \) are **left \( H \)-equivalent** if they produce the same left coset of \( H \),

\[
g \sim_L g' \quad \text{if} \quad gH = g'H.
\]

The verification that left \( H \)-equivalence is indeed an equivalence relation on \( G \) is straightforward. Thus left \( H \)-equivalence partitions \( G \) into disjoint equivalence classes, the left cosets,

\[
G = \bigsqcup gH \quad \text{(disjoint union of cosets, not union over all} \quad g \in G).\]

The **left coset space** is the set of left cosets,

\[
G/H = \{gH\} \quad \text{(each element of the set is itself a coset).}
\]

Also, one shows instantly that

\[
g \sim_L g' \iff g^{-1}g' \in H, \quad g, g' \in G.
\]

Naturally we could also define right \( H \)-equivalence and repeat the ideas,

\[
G = \bigsqcup Hg, \quad H\backslash G = \{Hg\}, \quad g \sim_R g' \iff g'g^{-1} \in H.
\]

Now, for any \( g \in G \) we have a bijection between \( H \) and \( gH \),

\[
H \leftrightarrow gH, \quad h \mapsto gh.
\]

And similarly for \( H \) and right cosets \( Hg \). Consequently all cosets have the same cardinality,

\[
|gH| = |Hg| = |H|, \quad g \in G.
\]

From the decompositions \( G = \bigsqcup gH = \bigsqcup Hg \) we then get

\[
|G| = |G/H| \cdot |H| = |H\backslash G| \cdot |H|.
\]

Define the **index** of \( H \) in \( G \) to be the shared cardinality of the coset spaces,

\[
\]

If \( G \) is finite then \([G : H]\) is a positive integer. But by the previous-but-first display,

\[
\]

And thus:

**Theorem 1.2 (Lagrange).** Let \( G \) be a finite group, and let \( H \) be a subgroup of \( G \). Then \(|H| \) divides \(|G|\).

Lagrange’s Theorem has many corollaries:

- If \( G \) is a prime-order group then it is cyclic.
- If \( G \) is a finite group and \( a \in G \) then \(|a| \mid |G|\).
• (Euler) Let \( n \in \mathbb{Z}_{>0} \). Then \( a^{\varphi(n)} \equiv 1 \mod n \) if \( \gcd(a, n) = 1 \).

• (Fermat) Let \( p \) be prime. Then \( a^{p-1} \equiv 1 \mod p \) if \( p \nmid a \).

2. Multiplicity of Indices

Let \( A \) be a supergroup of \( B \), in turn a supergroup of \( C \),

\[ C \subset B \subset A. \]

Thus

\[ A = \bigsqcup_{i} a_{i}B, \quad [A : B] = |\{a_{i}\}| \]

and

\[ B = \bigsqcup_{j} b_{j}C, \quad [B : C] = |\{b_{j}\}|. \]

Essentially immediately,

\[ A = \bigsqcup_{i,j} a_{i}b_{j}C. \]

Indeed, the union in the previous display is disjoint because if \( a_{i}'b_{j}'C = a_{i}b_{j}C \) then the cosets \( a_{i}'B \) and \( a_{i}B \) are nondisjoint, making them equal, so that \( a_{i}' = a_{i} \), and then we have \( b_{j}'C = b_{j}C \), giving \( b_{j}' = b_{j} \). Since the union is disjoint and the \((i, j)\)th coset contains the product \( a_{i}b_{j} \), no two such products are equal unless they involve the same \( a_{i} \) and the same \( b_{j} \). The multiplicativity of indices follows,

\[ [A : C] = |\{a_{i}b_{j}\}| = |\{a_{i}\}||\{b_{j}\}| = [A : B][B : C]. \]

3. Cosets and Actions

Consider a transitive action

\[ G \times S \rightarrow S. \]

Here \( G \) is a group, \( S \) is a set, and the action takes any point of \( S \) to any other.

Fix a point \( x \in S \), and let \( G_x \) be its isotropy subgroup,

\[ G_x = \{ g \in G : gx = x \}. \]

As we have discussed, \( G_x \) is indeed a subgroup of \( G \), but it need not be normal.

There is a natural set bijection between the resulting left coset space and the set,

\[ G / G_x \longleftrightarrow S, \quad gG_x \longleftrightarrow gx. \]

To see this, recall that \( G / G_x \) is the disjoint union of the left cosets,

\[ G = \bigsqcup gG_x, \]

and for any \( g, g' \in G \),

\[ g'G_x = gG_x \iff g^{-1}g' \in G_x \iff g^{-1}g'x = x \iff g'x = gx. \]

That is, each coset collectively moves \( x \) to a well-defined point of \( S \), and distinct cosets move \( x \) to distinct points.

For an example, let

\[ G = \text{SL}_2(\mathbb{R}), \quad S = \mathcal{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}. \]
Then $G$ acts on $S$ by the formula

$$
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix} (z) = \frac{az + b}{cz + d}.
$$

One key fact here is that

$$
\left( \begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix} \begin{bmatrix}
    a' & b' \\
    c' & d'
\end{bmatrix} \right) (z) = \begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix} \left( \begin{bmatrix}
    a' & b' \\
    c' & d'
\end{bmatrix} (z) \right),
$$

and another is that

$$
\text{Im} \left( \frac{az + b}{cz + d} \right) = \text{Im}(z) \frac{|cz + d|^2}{cz + d}.
$$

Now take our particular point to be $x = i$.

Then its isotropy groups is the 2-by-2 special orthogonal group,

$$
G_x = \left\{ \begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix} \in \text{SL}_2(\mathbb{R}) : \begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix} (i) = i \right\}
$$

$$
= \left\{ \begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix} \in \text{SL}_2(\mathbb{R}) \right\}
$$

$$
= \text{SO}(2).
$$

Thus the complex upper half plane has a completely real group-theoretic description as a coset space,

$$
\mathcal{H} \approx \text{SL}_2(\mathbb{R})/\text{SO}(2).
$$

No claim is being made here that the quotient space carries a group structure.