BASIC FACTS ABOUT GROUPS

(To be filled in later.)

• Uniqueness of identity: \( e' = e' e = e \).
• Uniqueness of inverse: \( b = b e = b(a e) = (b a)c = e c = e \).
• Granting a right-identity and right-inverses, they are two-sided: Given \( a \), let \( b \) a right-inverse of \( a \) and then let \( c \) be a right-inverse of \( b \). Then also \( b \) is a left-inverse of \( a \),

\[
ba = (ba)e = (ba)(bc) = ((ba)b)c = (b(ab))c = (be)c = be = e.
\]

Now compute that \( e \) is a left-identity since given \( a \),

\[
e a = (ab)a = a(ba) = ae = a.
\]
• Generalized associativity: For \( n \geq 2 \) let \( P(n) \) be the proposition that for all group elements \( g_1, \ldots, g_n \), all groupings of the product \( g_1 \cdots g_n \) are equal.

Certainly \( P(2) \) holds. And if \( n > 2 \) and \( P(k) \) holds for \( 2 \leq k < n \) then \( P(n) \) follows by generalized induction,

\[
\begin{align*}
(g_1 \cdots g_i)(g_{i+1} \cdots g_n) &= (g_1 \cdots g_i)((g_{i+1} \cdots g_j)(g_{j+1} \cdots g_n)) \\
&= (g_1 \cdots g_i)(g_{i+1} \cdots g_j)(g_{j+1} \cdots g_n) \\
&= (g_1 \cdots g_j)(g_{j+1} \cdots g_n).
\end{align*}
\]
• Generalized commutativity in abelian groups.
• \( g^2 = g \implies g = e \).
• Left and right cancellation laws.
• \( (g^{-1})^{-1} = g \) because \( g^{-1}g = e \).
• \( (ab)^{-1} = b^{-1}a^{-1} \).
• \( ax = b \iff x = a^{-1}b \) and \( xa = b \iff x = ba^{-1} \).
• For any \( a \in G \) and \( n \in \mathbb{Z} \), define

\[
a^n = \begin{cases} 
    e & \text{if } n = 0, \\
    a^{n-1} a & \text{if } n > 0, \\
    (a^{-n})^{-1} & \text{if } n < 0.
\end{cases}
\]

Then

\[
a^{n+m} = a^n a^m \quad \text{and} \quad (a^n)^m = a^{nm} \quad \text{for all } n, m \in \mathbb{Z}.
\]
• Definition of subgroup, various subgroups tests, arbitrary intersection of subgroups is a subgroup. Definition of \( \langle S \rangle \).