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Eight Lectures on Monomial Ideals

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Preface

The lectures below are an expanded form of the notes from a course given by Bernd Sturmfels in May, 1999 at the COCOA VI Summer School in Turin, Italy. They are not meant to be a complete overview of the latest research on monomial ideals. Rather, a few representative topics are presented in what we hope is their most accessible form, with lots of examples and pictures. Many proofs are omitted or only sketched, and often we avoid the most general form of a theorem. The reader who peruses these notes should acquire a feel for some of the recent developments in monomial ideals and surrounding areas, and be able to apply the theorems in practical cases, such as those in the exercises of Appendix A. At the very least, these lectures should provide a guide to the references, for those who wish to explore this exciting and branching field for themselves.

The lectures can be broken down roughly as follows. After the Basics (definitions and notation), Lectures I and II give what can be considered as a mostly historical overview. The standard classes of monomial ideals are introduced, including squarefree ("Stanley-Reisner") and Borel-fixed ideals, along with the ways of getting structural and numerical information about their resolutions and Hilbert series. Lecture III accomplishes the transition to recent advances by emphasizing the geometric ways of thinking about monomial ideals and their resolutions which have been so influential. Each of Lectures IV through VIII then treats in greater detail some recent research topic.

Lecture IV introduces a new paradigm for monomial ideals, contrasting with those of Lectures I and II. This is a notion of *genericity* which rests on randomness of exponents on monomials rather than of coefficients in polynomials. It is shown how the infinite poset of monomials in the ideal can be whittled down in this case to just a few absolutely necessary elements, and how this reduction reflects the geometry and combinatorics introduced in Lecture III. In particular, geometric free resolutions of generic monomial ideals (*algebraic Scarf complexes*) are constructed using just the combinatorial data. By deforming exponents on generators, any monomial ideal is approximated by a generic one, and bounds for Betti numbers of ideals are thus attained.

Lecture V abstracts the construction of Scarf complexes to more general *cellular resolutions*. In particular, the geometry of the Scarf complex is elucidated as a special case of the *hull resolution* defined by convexity. This lecture shows how combinatorial topology interacts with monomial ideals via cellular complexes which are not necessarily simplicial.

Lecture VI then covers Alexander duality, illustrating how this topological duality is manifested in monomial ideals and their homological algebra. First the classical form for squarefree monomial ideals is reviewed, along with some recent consequences for resolutions of these. Then, Alexander duality is defined for arbitrary monomial ideals. The interactions with free resolutions are explored, including what happens to the numerical information in general, and in particular how the geometry of cellular resolutions demonstrates the duality. Cogeneric monomial ideals, which are generic with respect to their irreducible components rather than their generators, are presented as running examples.

Lecture VII connects monomial and binomial ideals via monomial modules. Certain kinds of binomial ideals—the *lattice ideals*, which include toric ideals—are viewed as coming from "infinite periodic" versions of monomial ideals. The cellular methods of Lecture V apply here as well, with the cell complexes having infinitely many cells but finite dimension. Taking the quotient by periodicity, the cell complex becomes a torus, and resolves the lattice ideal. Examples of the theory are provided by the classes of *unimodular Lawrence ideals* and *generic lattice ideals.*

Finally, Lecture VIII relates recent advances in the study of local cohomology for monomial ideals to the foundational theorem of Hochster on the subject. The standard notion of local cohomology with supports is reviewed, and calculations of Hilbert series as well as module structure are given. Using the *Čech hull*, it is then shown how these calculations are equivalent to the classical ones.

Acknowledgments. The authors are greatly indebted to Bernd Sturmfels: these lectures were, after all, originally delivered by him, and he provided many comments on drafts along the way. Those who attended the summer school will see his influence throughout, including the exercises in Appendix A. We also wish to thank the students of the Summer School; in particular, many of the solutions in Appendix B are adapted from students' responses solicited for these notes. Finally, we would like to thank the organizers of the COCOA conference and school, Tony Geramita, Lorenzo Robbiano, Vincenzo Ancona, and Alberto Conte for inviting us to Italy; our thanks especially to Tony and Lorenzo, for initiating this written project.

0 Basics

Here we introduce the objects and notation surrounding monomial ideals. Some of this material is a little technical, and most of it will be review for many readers, who should proceed to the main Lectures and refer back as necessary.

0.1 $\mathbb{Z}^n$-grading

Let k be a field and $S := k[\mathbf{x}] := k[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates. A *monomial* in $k[\mathbf{x}]$ is a product $\mathbf{x}^\mathbf{a} := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ for a vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ of nonnegative integers. An ideal $I \subseteq k[\mathbf{x}]$ is called a *monomial ideal* if it is generated by monomials. A polynomial f is in a monomial ideal $I = \langle \mathbf{x}^{\mathbf{a_1}}, \ldots, \mathbf{x}^{\mathbf{a_r}} \rangle$ if and only if each term of f is divisible by one of the given generators $\mathbf{x}^{\mathbf{a_i}}$. It follows that a monomial ideal has a unique minimal set of monomial generators, and this set is finite by the Hilbert Basis Theorem.

As a k-vector space, the polynomial ring S is a direct sum $S = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S_\mathbf{a}$, where $S_\mathbf{a}$ is the k-span of the monomial $\mathbf{x}^\mathbf{a}$. Since $S_\mathbf{a} \cdot S_\mathbf{b} \subseteq S_{\mathbf{a}+\mathbf{b}}$, we say that S is an $\mathbb{N}^n$*-graded* k-algebra. More generally, an S-module M is said to be $\mathbb{Z}^n$*-graded* if $M = \bigoplus_{\mathbf{b} \in \mathbb{Z}^n} M_\mathbf{b}$ is a direct sum of k-vector spaces with $S_\mathbf{a} \cdot M_\mathbf{b} \subseteq M_{\mathbf{a}+\mathbf{b}}$.

Example 0.1 The following are all $\mathbb{Z}^n$-graded S-modules:

1. Monomial ideals $I = \bigoplus_{\mathbf{x}^\mathbf{a} \in I} S_\mathbf{a}$, and quotients $S/I = \bigoplus_{\mathbf{x}^\mathbf{a} \notin I} S_\mathbf{a}$.
2. The *Laurent polynomial module* $T = S[x_1^{-1}, \ldots, x_n^{-1}] = \bigoplus_{\mathbf{b} \in \mathbb{Z}^n} \mathbf{x}^\mathbf{b}$. Here, $\mathbf{x}^\mathbf{b}$ is called a *Laurent monomial* since the exponent may have negative coordinates. This module will be important later on, when we discuss monomial modules (Lecture VII).

3. The localization $S[x_i^{-1}]$ is $\mathbb{Z}^n$-graded, its nonzero components being 1-dimensional in every degree $\mathbf{b}$ such that all coordinates are nonnegative except for possibly b_i. More generally, if $\sigma \subseteq \{1, \ldots, n\}$, the localization

$$S[\mathbf{x}^{-\sigma}] \quad := \quad S[x_i^{-1} \mid i \in \sigma] \quad = \bigoplus_{b_j \geq 0 \text{ if } j \notin \sigma} k \cdot \mathbf{x}^{\mathbf{b}}$$

is $\mathbb{Z}^n$-graded. Throughout these lectures, $\mathbf{x}^\sigma = \prod_{i \in \sigma} x_i$ for $\sigma \subseteq \{1, \ldots, n\}$. □

Given a $\mathbb{Z}^n$-graded module M, the *$\mathbb{Z}^n$-graded shift* $M[\mathbf{a}]$ for $\mathbf{a} \in \mathbb{Z}^n$ is the $\mathbb{Z}^n$-graded module defined by $M[\mathbf{a}]_{\mathbf{b}} = M_{\mathbf{a}+\mathbf{b}}$. In particular, the free S-module of rank one generated in degree $\mathbf{a}$ is $S[-\mathbf{a}]$. We will sometimes denote the element $1 \in S[-\mathbf{a}]_{\mathbf{a}}$ by $1_{\mathbf{a}}$; thus we can write $x^{\mathbf{b}} \cdot 1_{\mathbf{a}} \in S[-\mathbf{a}]_{\mathbf{b}+\mathbf{a}}$. If $\mathbf{a} \in \mathbb{N}^n$, sending $1_{\mathbf{a}}$ to $\mathbf{x}^{\mathbf{a}}$ induces a $\mathbb{Z}^n$-graded S-module isomorphism between $S[-\mathbf{a}]$ and the principal ideal $\langle \mathbf{x}^{\mathbf{a}} \rangle \subset S$.

If N is another $\mathbb{Z}^n$-graded module, then $N \otimes_S M$ is $\mathbb{Z}^n$-graded, with degree $\mathbf{c}$ component $(N \otimes_S M)_{\mathbf{c}}$ generated by all elements $n_{\mathbf{a}} \otimes m_{\mathbf{b}}$ with $n_{\mathbf{a}} \in N_{\mathbf{a}}$ and $m_{\mathbf{b}} \in M_{\mathbf{b}}$ such that $\mathbf{a} + \mathbf{b} = \mathbf{c}$. For example, $S[\mathbf{a}] \otimes_S M = M[\mathbf{a}]$, with degree $\mathbf{b}$ component $M[\mathbf{a}]_{\mathbf{b}} = 1_{-\mathbf{a}} \otimes M_{\mathbf{a}+\mathbf{b}}$.

Every $\mathbb{Z}^n$-graded module M is also $\mathbb{Z}$-graded, with $M_d = \bigoplus_{|\mathbf{a}|=d} M_{\mathbf{a}}$. This transition is sometimes called “passing from the *fine* to the *coarse* grading”.

0.2 Monomial matrices

A homomorphism $\phi : M \to N$ of $\mathbb{Z}^n$-graded modules is, unless otherwise stated, required to be *of degree* $\mathbf{0}$; that is, $\phi(M_{\mathbf{b}}) \subseteq N_{\mathbf{b}}$ for all $\mathbf{b} \in \mathbb{Z}^n$. For instance, if $M = S[-\mathbf{a}_M]$ and $N = S[-\mathbf{b}_N]$ are free modules generated in degrees $\mathbf{a}_M$ and $\mathbf{a}_N$, then there exists a nonzero homomorphism of degree $\mathbf{0}$ if and only if $\mathbf{a}_M \succeq \mathbf{a}_N$. (The “$\succeq$” symbol is used to denote the partial order on $\mathbb{Z}^n$ in which $\mathbf{a} \succeq \mathbf{b}$ if $a_i \geq b_i$ for all $i \in \{1, \ldots, n\}$.) This is because the generator of M has to map to a nonzero element of N in degree $\mathbf{a}_M$. In fact we can map the basis element of $S[-\mathbf{a}_M]$ to any element in $N_{\mathbf{a}_M}$, so

$$\operatorname{Hom}(S[-\mathbf{a}_M], S[-\mathbf{a}_N]) = S[-\mathbf{a}_N]_{\mathbf{a}_M} = \begin{cases} k & \text{if } \mathbf{a}_M \succeq \mathbf{a}_N \\ 0 & \text{if } \mathbf{a}_M \not\succeq \mathbf{a}_N \end{cases}.$$

Therefore, if we want to write down a $\mathbb{Z}^n$-graded map $S[-\mathbf{a}_M] \to S[-\mathbf{a}_N]$, we only have to specify a constant $\lambda \in k$, with the stipulation that $\lambda = 0$ unless $\mathbf{a}_M \succeq \mathbf{a}_N$.

More generally, if $M = \bigoplus_p S[-\mathbf{a}_p]$ and $N = \bigoplus_q S[-\mathbf{a}_q]$ are arbitrary $\mathbb{Z}^n$-graded free modules, then a map $M \to N$ can be specified by a matrix with entries $\lambda_{qp} \in k$. But we also have to remember the degrees $\mathbf{a}_p$ and $\mathbf{a}_q$ in the source and target. To do this, we make the following

Definition 0.2 A *monomial matrix* is a matrix of constants λ_{qp} whose columns are labeled by the source degrees $\mathbf{a}_p$ and whose rows are labeled by the target degrees $\mathbf{a}_q$, and such that $\lambda_{qp} = 0$ unless $\mathbf{a}_p \succeq \mathbf{a}_q$.

The general monomial matrix therefore represents a map that looks like

$$\begin{array}{ccc} & \cdots\ \mathbf{a}_p\ \cdots & \\ & \begin{array}{c}\vdots\\ \mathbf{a}_q\\ \vdots\end{array}\left(\begin{array}{ccc} & & \\ & \lambda_{qp} & \\ & & \end{array}\right) & \\ \bigoplus_q S[-\mathbf{a}_q] & \xleftarrow{\qquad\qquad\qquad} & \bigoplus_p S[-\mathbf{a}_p]\,. \end{array}$$

Sometimes we label the rows and columns with monomials $\mathbf{x}^{\mathbf{a}}$ instead of vectors $\mathbf{a}$.

Each $\mathbb{Z}^n$-graded free module can also be regarded as an ungraded free module, and most readers will have seen already matrices used for maps of (ungraded) free modules over arbitrary rings. In order to recover the more usual notation, simply replace each matrix entry λ_{qp} by $\mathbf{x}^{\mathbf{a}_p-\mathbf{a}_q}\lambda_{qp}$, and then forget the border row and column. Because of the conditions defining monomial matrices, $\mathbf{x}^{\mathbf{a}_p-\mathbf{a}_q}\lambda_{qp} \in S$ for all p and q.

0.3 Complexes and resolutions

A *homological complex* of S-modules is a sequence $\cdots \stackrel{\phi_{i-1}}{\longleftarrow} F_{i-1} \stackrel{\phi_i}{\longleftarrow} F_i \longleftarrow \cdots$ of S-module homomorphisms such that $\phi_{i-1}\circ\phi_i = 0$. In most of the examples from these lectures, the modules will be $\mathbb{Z}^n$-graded and the maps homogeneous of degree $\mathbf{0}$. A complex is *exact at the* i^{th} *step* if it has no homology there; that is, if kernel$(\phi_{i-1}) =$ image(ϕ_i). The complex is *exact* if it is exact at the i^{th} step for all $i \in \mathbb{Z}$.

A *free resolution* of an S-module M is a complex

$$0 \longleftarrow F_0 \stackrel{\phi_1}{\longleftarrow} F_1 \longleftarrow \cdots \longleftarrow F_{t-1} \stackrel{\phi_t}{\longleftarrow} F_t \longleftarrow 0$$

of free S-modules which is exact everywhere except the 0^{th} step, and such that $M = \text{coker}(\phi_1) = F_0/\text{image}(\phi_1)$. Sometimes we *augment* the free resolution with the surjection $0 \leftarrow M \stackrel{\phi_0}{\longleftarrow} F_0$, to make the complex exact everywhere. The length of the resolution, by definition the greatest homological degree of a nonzero module in the resolution ($= t$, assuming $F_t \neq 0$), is called the *projective dimension of* M.

Every S-module has a free resolution, with length $\leq n$. If M is $\mathbb{Z}^n$-graded, then it has a $\mathbb{Z}^n$-graded free resolution. If, in addition, M is finitely generated, there is a $\mathbb{Z}^n$-graded resolution $M \leftarrow \mathbb{F}_{\bullet}$ in which all of the ranks of the F_i are finite and simultaneously minimized. Such an $\mathbb{F}_{\bullet}$ is called a *minimal free resolution* of M, and is unique up to noncanonical isomorphism (see [12, Theorem 20.2 and Exercise 20.1]).

In any $\mathbb{Z}^n$-graded free resolution resolution, the i^{th} term F_i has a direct sum decomposition

$$F_i = \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} S[-\mathbf{a}]^{\beta_{i,\mathbf{a}}}.$$

Furthermore, we can write down each map ϕ_i using a monomial matrix. By definition, the top border row (source degrees) $\mathbf{a}_p$ on a monomial matrix for ϕ_i equal the left border column (target degrees) $\mathbf{a}_q$ on a monomial matrix for ϕ_{i+1}. (See Example 1.5

in Section 1.1 for illustration.) The minimal free resolution is then characterized by having scalar entry $\lambda_{qp} = 0$ whenever $\mathbf{a}_p = \mathbf{a}_q$ in any of its monomial matrices. Note that if the monomial matrices are made ungraded as in the previous section, this simply means that the nonzero entries in the matrices are nonconstant monomials (with coefficients), and agrees with the usual notion of minimality for $\mathbb{Z}$-graded resolutions.

In the case of a $\mathbb{Z}^n$-graded minimal free resolution of M, the number $\beta_{i,\mathbf{a}}(M) := \beta_{i,\mathbf{a}}$ is an invariant of the finitely generated module M called the i-th *Betti number of M in degree* $\mathbf{a}$. It is readily seen that this number is the dimension of the degree $\mathbf{a}$ piece of a Tor module:

$$\beta_{i,\mathbf{a}}(M) = \dim_k(\mathrm{Tor}_i^S(k, M)_{\mathbf{a}}) .$$

Indeed, tensoring a minimal free resolution of M with $k = S/\langle x_1, \ldots, x_n \rangle$ turns all of the ϕ_i into zero-maps. These Tor modules are the same ones we use to measure the usual $\mathbb{Z}$-graded Betti numbers. Therefore, the multigraded Betti numbers are more refined, in the sense that the $\mathbb{Z}$-graded Betti numbers can be obtained from them: $\beta_{i,d}(M) = \sum_{|\mathbf{a}|=d} \beta_{i,\mathbf{a}}(M)$, where $|\mathbf{a}| = \sum_j a_j$. Of course, this can be seen directly from the minimal free resolution itself, which is already $\mathbb{Z}$-graded.

0.4 Hilbert series

Let M be a $\mathbb{Z}^n$-graded module such that $\dim_k(M_{\mathbf{a}})$ is finite for all $\mathbf{a} \in \mathbb{Z}^n$. The (finely graded) *Hilbert series* of M is the formal power series

$$H(M; \mathbf{x}) := H(M; x_1, \ldots, x_n) := \sum_{\mathbf{a} \in \mathbb{Z}^n} \dim_k(M_{\mathbf{a}}) \cdot \mathbf{x}^{\mathbf{a}} .$$

For example,

$$H(S; \mathbf{x}) = \prod_{i=1}^{n} \frac{1}{1 - x_i} \;\; = \text{sum of all monomials in } S ,$$

and

$$H(S[-\mathbf{a}]; \mathbf{x}) = \frac{\mathbf{x}^{\mathbf{a}}}{\prod_{i=1}^{n}(1 - x_i)}$$

for $\mathbf{a} \in \mathbb{Z}^n$. Of course, the primary example for us will be

$$H(S/I; \mathbf{x}) = \text{sum of all monomials not in } I ,$$

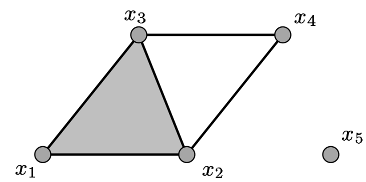
where I is a monomial ideal. For those more accustomed to $\mathbb{Z}$-graded modules, the usual (coarse) Hilbert series $H(M; t, \ldots, t)$ is obtained by substituting $x_i = t$ for all i.

Given a short exact sequence $0 \leftarrow M'' \leftarrow M \leftarrow M' \leftarrow 0$, the rank-nullity theorem from linear algebra implies that $\dim_k(M_{\mathbf{a}}) = \dim_k(M''_{\mathbf{a}}) + \dim_k(M'_{\mathbf{a}})$ for all $\mathbf{a}$, and hence $H(M; \mathbf{x}) = H(M''; \mathbf{x}) + H(M'; \mathbf{x})$. More generally, if $0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$ is a finite exact sequence such as a free resolution, then $H(M; \mathbf{x}) = \sum_i (-1)^i H(F_i; \mathbf{x})$. In particular, if M is finitely generated, the existence of a finite-rank free resolution for M implies that the Hilbert series of M is a rational function of $\mathbf{x}$, because it is an alternating sum of Hilbert series of $S[-\mathbf{a}]$ for various $\mathbf{a}$. Moreover, the denominator can always be taken to be $\prod_i (1 - x_i)$. A running theme of these notes is to analyze the numerator of the Hilbert series $H(S/I; \mathbf{x})$ for monomial ideals I.

0.5 Simplicial complexes and homology

An (abstract) *simplicial complex* Δ on $\{1, 2, \ldots, n\}$ is a collection of subsets of $\{1, \ldots, n\}$, closed under the operation of taking subsets. We frequently identify $\{1, \ldots, n\}$ with the variables $\{x_1, \ldots, x_n\}$, as in Example 0.3, below. An element of a simplicial complex is called a *face* or *simplex*. A simplex $\sigma \in \Delta$ of cardinality $i+1$ is called an *i-dimensional face* or an *i-face* of Δ. The empty set, $\varnothing$, is the unique face of dimension -1, as long as Δ is not the *void complex* $\{\}$ consisting of no subsets of $\{1, \ldots, n\}$ (which has no faces at all). The *dimension* of Δ, denoted $\dim(\Delta)$, is defined to be the maximum of the dimensions of its faces (or $-\infty$ if $\Delta = \{\}$).

Example 0.3 The simplicial complex Δ on $\{1, 2, 3, 4, 5\}$ consisting of all subsets of the sets $\{1, 2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, and $\{5\}$ is pictured below:



The simplicial complex Δ

Note that Δ is completely specified by its *facets*, or maximal faces, by definition of simplicial complex. □

Let Δ be a simplicial complex on $\{1, \ldots, n\}$. For $i \in \mathbb{Z}$, let $F_i(\Delta)$ be the set of i-dimensional faces of Δ, and let $k^{F_i(\Delta)}$ be a k-vector space whose basis elements e_σ correspond to the i-faces $\sigma \in F_i(\Delta)$. The *(augmented* or *reduced) chain complex of* Δ *over* k is the complex

$$\widetilde{\mathcal{C}}_\bullet(\Delta; k): \quad 0 \longleftarrow k^{F_{-1}(\Delta)} \xleftarrow{\partial_0} \cdots \longleftarrow k^{F_{i-1}(\Delta)} \xleftarrow{\partial_i} k^{F_i(\Delta)} \longleftarrow \cdots \xleftarrow{\partial_{n-1}} k^{F_{n-1}(\Delta)} \longleftarrow 0\,,$$

where for an i-face σ,

$$\partial_i(e_\sigma) = \sum_{j \in \sigma} \operatorname{sign}(j, \sigma)\, e_{\sigma \setminus \{j\}}\,.$$

Here, $\operatorname{sign}(j, \sigma) = (-1)^{r-1}$ if j is the r^{th} element of the set σ, written in increasing order. If $i < -1$ or $n - 1 < i$, then $k^{F_i(\Delta)} = 0$ and $\partial_i = 0$ by definition. The reader unfamiliar with simplicial complexes should make the routine check that $\partial_i \circ \partial_{i+1} = 0$. For $i \in \mathbb{Z}$, the k-vector space

$$\widetilde{H}_i(\Delta; k) := \operatorname{kernel}(\partial_i)/\operatorname{image}(\partial_{i+1})$$

is the *i-th reduced homology* of Δ over k. In particular, $\widetilde{H}_{n-1}(\Delta; k) = \operatorname{kernel}(\partial_{n-1})$ and $\widetilde{H}_i(\Delta; k) = 0$ for $i < 0$ or $n-1 < i$, unless $\Delta = \{\varnothing\}$, in which case $\widetilde{H}_{-1}(\Delta; k) \cong k$ and $\widetilde{H}_i(\Delta; k) = 0$ for $i \geq 0$. The dimension of $\widetilde{H}_0(\Delta; k)$ as a k-vector space is one less than the number of connected components of Δ. Elements of $\operatorname{kernel}(\partial_i)$ are called *i-cycles* and elements of $\operatorname{image}(\partial_{i+1})$ are called *i-boundaries*.

Example 0.4 For Δ as in Example 0.3, we have

$$\begin{aligned} F_2(\Delta) &= \{\{1,2,3\}\}, \\ F_1(\Delta) &= \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\}\}, \\ F_0(\Delta) &= \{\{1\},\{2\},\{3\},\{4\},\{5\}\}, \\ F_{-1}(\Delta) &= \{\varnothing\}. \end{aligned}$$

Choosing bases for the $k^{F_i(\Delta)}$ as suggested by the ordering of the faces listed above, the chain complex for Δ becomes

$$0 \longleftarrow k \xleftarrow[\partial_0]{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}} k^5 \xleftarrow[\partial_1]{\begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}} k^5 \xleftarrow[\partial_2]{\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}} k \longleftarrow 0\,.$$

For example, $\partial_2(e_{\{1,2,3\}}) = e_{\{2,3\}} - e_{\{1,3\}} + e_{\{1,2\}}$, which we identify with the vector $(1,-1,1,0,0)$. The mapping ∂_1 has rank 3, so $\widetilde{H}_0(\Delta;k) \cong \widetilde{H}_1(\Delta;k) \cong k$ and the other homology groups are 0. Geometrically, $\widetilde{H}_0(\Delta;k)$ is nontrivial since Δ is disconnected and $\widetilde{H}_1(\Delta;k)$ is nontrivial since Δ contains a triangle which is not the boundary of an element of Δ. □

Remark 0.5 We wouldn't make such a big deal about the difference between the empty complex $\{\varnothing\}$ and the void complex $\{\}$ if it didn't come up so much. Many of the formulas for Betti numbers, dimensions of local cohomology, and so on depend on the fact that $\widetilde{H}_i(\{\varnothing\};k)$ is nonzero for $i=-1$, while $\widetilde{H}_i(\{\};k) = 0$ for all i. □

The *(augmented* or *reduced) cochain complex of* Δ *over* k is the k-dual $\widetilde{\mathcal{C}}^{\bullet}(\Delta;k) = \mathrm{Hom}_k(\widetilde{\mathcal{C}}_{\bullet}(\Delta;k),k)$ of the chain complex. Explicitly, let $k^{F_i^*(\Delta)} = \mathrm{Hom}_k(k^{F_i(\Delta)},k)$ have basis $\{e_\sigma^* \mid \sigma \in F_i(\Delta)\}$ dual to the basis of $k^{F_i(\Delta)}$. Then

$$\widetilde{\mathcal{C}}^{\bullet}(\Delta;k): \quad 0 \longrightarrow k^{F^*_{-1}(\Delta)} \xrightarrow{\partial^0} \cdots \longrightarrow k^{F^*_{i-1}(\Delta)} \xrightarrow{\partial^i} k^{F^*_i(\Delta)} \longrightarrow \cdots \xrightarrow{\partial^{n-1}} k^{F^*_{n-1}(\Delta)} \longrightarrow 0\,,$$

is the cochain complex of Δ, where for an $(i-1)$-face σ,

$$\partial^i(e_\sigma^*) = \sum_{\substack{j \notin \sigma \\ j \cup \sigma \in \Delta}} \mathrm{sign}(j, \sigma \cup j)\, e^*_{\sigma \cup \{j\}}$$

is the transpose of ∂_i.

Example 0.6 The cochain complex for Δ as in the previous two examples is exactly the same as the complex in Example 0.4, except that the arrows should be reversed and the elements of the vector spaces should be considered as row vectors, with the matrices acting by multiplication on the right. □

For $i \in \mathbb{Z}$, the k-vector space

$$\widetilde{H}^i(\Delta; k) := \text{kernel}(\partial^{i+1})/\text{image}(\partial^i)$$

is the *i-th reduced cohomology* of Δ over k. Because $\text{Hom}_k(-, k)$ is exact, there is a canonical isomorphism $\widetilde{H}^i(\Delta; k) = \widetilde{H}_i(\Delta; k)^*$, where * denotes k-dual. Elements of kernel(∂^{i+1}) are called *i-cocycles* and elements of image(∂^i) are called *i-coboundaries.*

0.6 Irreducible decomposition

An arbitrary ideal in S is called *irreducible* if it is not the intersection of two strictly larger ideals. For example, prime ideals are irreducible. The standard noetherian argument shows that every ideal $I \subset k[x_1, \ldots, x_n]$ can be written as an intersection $Q_1 \cap \cdots \cap Q_r$ of irreducible ideals. Such intersections are of course not unique—it might be that intersecting all but one of the Q_i still yields I. Even assuming this is not so, i.e. that the intersection is *irredundant*, the irreducible decomposition still need not be unique. However, an irredundant decomposition of a monomial ideal as an intersection of irreducible monomial ideals is unique. Although it is elementary (but cumbersome) to prove this uniqueness directly, it will follow from the uniqueness of minimal monomial generating sets along with Alexander duality, which we treat in Lecture VI. But since irreducible decompositions come up many times before then, we describe now what they look like, and how to find them.

Here's a fun algorithm to produce irreducible decompositions: if m is a minimal generator of a monomial ideal I and $m = m'm''$ where m' and m'' are relatively prime monomials, then $I = (I + \langle m' \rangle) \cap (I + \langle m'' \rangle)$. Thus, for example,

$$I = \langle x_1 x_2^2, x_3 \rangle = (I + \langle x_1 \rangle) \cap (I + \langle x_2^2 \rangle) = \langle x_1, x_3 \rangle \cap \langle x_2^2, x_3 \rangle.$$

Iterating this process, it is clear that every monomial ideal can be expressed as an intersection of ideals generated by powers of some of the variables. It turns out, in fact, that these are irreducible. Therefore, an irreducible monomial ideal is uniquely determined by a vector $\mathbf{a} \in \mathbb{N}^n$, just like a monomial: we write $\mathfrak{m}^{\mathbf{a}} = \langle x_i^{a_i} \mid a_i \geq 1 \rangle$. This takes the monomial $\mathbf{x}^{\mathbf{a}}$ and sticks commas between the variables, ignoring those variables with exponent zero. If $\sigma \subseteq \{1, \ldots, n\}$, then $\mathfrak{m}^\sigma = \langle x_i \mid i \in \sigma \rangle$ denotes a monomial prime ideal. We use the symbol $\mathfrak{m}$ for irreducible ideals because it is commonly used to denote the maximal ideal $\langle x_1, \ldots, x_n \rangle$.

Further reading. The standard reference for $\mathbb{Z}^n$-graded modules is the paper by and Goto and Watanabe [16]. Monomial matrices are defined in [24]. Information on free resolutions can be found in Eisenbud's comprehensive book on commutative algebra [12], but a quicker introduction with monomial ideals partially in mind can be found in Chapter 0 of Stanley's green book [32], which is also an excellent reference for simplicial complexes. If you haven't had enough of irreducible decompositions by the time you finish these lectures, look at [23].

1 Lecture I: Squarefree monomial ideals

1.1 Equivalent descriptions

A monomial $\mathbf{x}^{\mathbf{a}}$ is called *squarefree* if every coordinate of $\mathbf{a}$ is 0 or 1. A monomial ideal is called *squarefree* if it is generated by squarefree monomials. The information carried by squarefree monomial ideals can be characterized in many ways. Some of the most important are given in the following

Theorem 1.1 *The following are equivalent:*

1. *Squarefree monomial ideals in* $k[x_1, \ldots, x_n]$
2. *Unions of coordinate subspaces in* k^n
3. *Unions of coordinate subspaces in* $\mathbf{P}^{n-1}$
4. *Simplicial complexes on* $\{1, \ldots, n\} := \{1, 2, \ldots, n\}$

The ideal $I = I_\Delta$ *in (1) is called the* face ideal *or* Stanley-Reisner ideal *of the simplicial complex* Δ *in (4), and* S/I_Δ *is called the* face ring *or* Stanley-Reisner ring *of* Δ.

Proof: The general idea is

$$\text{ideal } I \leftrightsquigarrow \text{affine variety of } I \leftrightsquigarrow \text{projective variety of } I$$
$$\text{and}$$
$$\text{coordinate subspace} \leftrightsquigarrow \text{simplex}\,.$$

Precisely:

$1 \rightsquigarrow 2$: Given a squarefree monomial ideal I, let $V(I) \subset k^n$ be the algebraic subset on which it vanishes. From the algorithm in Section 0.6 it follows that $I = \mathfrak{m}^{\sigma_1} \cap \cdots \cap \mathfrak{m}^{\sigma_r}$ is an intersection of monomial prime ideals. Thus $V(I) = V(\mathfrak{m}^{\sigma_1}) \cup \cdots \cup V(\mathfrak{m}^{\sigma_r})$ is a union of coordinate subspaces because $V(\mathfrak{m}^{\sigma})$ is the vector subspace of k^n spanned by the standard basis vectors $\{\mathbf{e}_j \mid j \notin \sigma\}$.

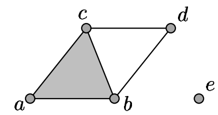
$2 \rightsquigarrow 3$: Recall that $\mathbb{P}^{n-1} = (k^n \setminus \{\mathbf{0}\})/k^*$ is a quotient of k^n. In particular, the quotient of any nonzero coordinate subspace of k^n is a coordinate subspace of $\mathbb{P}^{n-1}$. This gives a 1-1 correspondence between affine and projective coordinate subspaces if we agree that the empty set is a projective coordinate subspace, spanned by the empty set of coordinate points, corresponding to the affine linear subspace $\{\mathbf{0}\}$.

$3 \rightsquigarrow 4$: The simplicial complex on $\{1, \ldots, n\}$ corresponding to a union of coordinate subspaces is the one whose faces consist of those sets $\sigma \subseteq \{1, \ldots, n\}$ such that $\text{span}(\mathbf{e}_i \mid i \in \sigma)$ is contained in the union. It is a simplicial complex because if a subspace V is contained in our union, then so is every subspace of V.

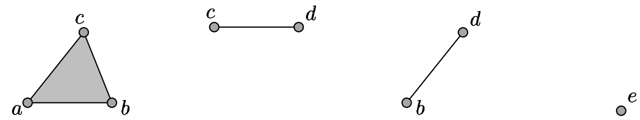
$4 \rightsquigarrow 3 \rightsquigarrow 2 \rightsquigarrow 1$: The algebraic sets in (3) and (2) are the unions of the subspaces $\text{span}(\mathbf{e}_i \mid i \in \sigma)$ for which σ is in our simplicial complex. Each coordinate subspace $\text{span}(\mathbf{e}_i \mid i \in \sigma)$ is equal to $V(\mathfrak{m}^{\overline{\sigma}})$, where $\overline{\sigma} := \{1, \ldots, n\} \setminus \sigma$, and the ideal in (1) is the intersection of these $\mathfrak{m}^{\overline{\sigma}}$. □

Remark 1.2 A little bit of caution is warranted: in $4 \rightsquigarrow 3 \rightsquigarrow 2 \rightsquigarrow 1$, it is not always true that the ideal of polynomials vanishing on a collection of coordinate subspaces is a monomial ideal! This means that the correspondence in the Theorem is *not* the

Zariski correspondence: there is a problem if k is finite. On the other hand, when k is infinite, the Zariski correspondence between ideals and algebraic sets does induce the 1-1 correspondence between squarefree monomial ideals and unions of coordinate subspaces. At any rate, the Theorem does not depend on k being infinite. $\square$

Example 1.3 The simplicial complex $\Delta =$ 

from Example 0.3, replacing the variables x_1, x_2, x_3, x_4, x_5 by a, b, c, d, e, has Stanley-Reisner ideal



$$\begin{aligned} I_\Delta &= \langle d, e\rangle \cap \langle a, b, e\rangle \cap \langle a, c, e\rangle \cap \langle a, b, c, d\rangle \\ &= \langle ad, ae, bcd, be, ce, de\rangle. \end{aligned}$$

We have expressed I_Δ via its irreducible decomposition and its minimal generators. Above each irreducible component is drawn the corresponding facet of Δ. $\square$

Given a simplicial complex Δ, it should be clear by now what the irreducible decomposition of its Stanley-Reisner ideal I_Δ means: each irreducible component $\mathfrak{m}^\sigma$ corresponds to a facet $\overline{\sigma} := \{1, \ldots, n\} \setminus \sigma$ of Δ. But what about the minimal generators? In order for $\mathbf{x}^\tau$ to be in the intersection $I_\Delta = \bigcap_{i=1}^r \mathfrak{m}^{\sigma_i}$, it is necessary and sufficient that τ share at least one element with σ_i for each i between 1 and r. In terms of Δ, this means that for every facet $\overline{\sigma} \in \Delta$, τ has at least one vertex not in $\overline{\sigma}$. In other words, τ is not contained in any facet (and therefore in any face) of Δ; we say that τ is a *nonface* of Δ. Thus the squarefree monomials in I_Δ correspond precisely to the nonfaces of Δ:

$$I_\Delta = \langle \mathbf{x}^\tau \mid \tau \notin \Delta \rangle .$$

Since being a nonface is preserved under taking supersets, the minimal generators of I_Δ are therefore $\{\mathbf{x}^\tau \mid \tau \text{ is a minimal nonface of } \Delta\}$. Alternatively, the nonzero squarefree monomials of S/I_Δ correspond to the faces of Δ. Example 1.3 can be used as a test case.

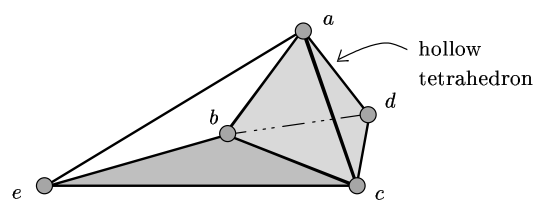
1.2 Hilbert series

Now we want to write down the Hilbert series of S/I_Δ for the Stanley-Reisner ring of a simplicial complex Δ. We know already from the previous section which *squarefree* monomials are not in I_Δ. But because the generators of I_Δ are themselves squarefree, a monomial $\mathbf{x}^\mathbf{a}$ is not in I_Δ if and only if $\mathbf{x}^{\mathrm{supp}(\mathbf{a})}$ is not in I_Δ, where $\mathrm{supp}(\mathbf{a}) = \{i \in \{1, \ldots, n\} \mid a_i \neq 0\}$ is the *support* of $\mathbf{a}$. Therefore,

$$H(S/I_\Delta; x_1, \ldots, x_n) = \sum \{\mathbf{x}^\mathbf{a} \mid \mathbf{a} \in \mathbb{N}^n \text{ and } \mathrm{supp}(\mathbf{a}) \in \Delta\}$$

$$
\begin{aligned}
&= \sum_{\sigma\in\Delta} \sum \{\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{N}^n \text{ and } \operatorname{supp}(\mathbf{a}) = \sigma\} \\
&= \sum_{\sigma\in\Delta} \prod_{i\in\sigma} \frac{x_i}{1-x_i} \\
&= \frac{1}{(1-x_1)\cdots(1-x_n)} \cdot \underbrace{\left\{ \sum_{\sigma\in\Delta} \prod_{i\in\sigma} x_i \cdot \prod_{j\notin\sigma} (1-x_j) \right\}}_{\text{numerator of the Hilbert series}}.
\end{aligned}
$$

Example 1.4 Consider the simplicial complex Γ depicted below. (The reason for not calling it Δ is because Γ is the Alexander dual of the simplicial complex Δ of Examples 0.3 and 1.3, and calling them both Δ would be confusing in Lecture VI.)



The simplicial complex Γ

The Stanley-Reisner ideal of Γ is

$$
\begin{aligned}
I_\Gamma &= \langle de, abe, ace, abcd \rangle \\
&= \langle a,d\rangle \cap \langle a,e\rangle \cap \langle b,c,d\rangle \cap \langle b,e\rangle \cap \langle c,e\rangle \cap \langle d,e\rangle,
\end{aligned}
$$

and its Hilbert series is

$$
\begin{aligned}
&1 + \tfrac{a}{1-a} + \tfrac{b}{1-b} + \tfrac{c}{1-c} + \tfrac{d}{1-d} + \tfrac{e}{1-e} + \tfrac{ab}{(1-a)(1-b)} + \tfrac{ac}{(1-a)(1-c)} + \tfrac{ad}{(1-a)(1-d)} \\
&+ \tfrac{ae}{(1-a)(1-e)} + \tfrac{bc}{(1-b)(1-c)} + \tfrac{bd}{(1-b)(1-d)} + \tfrac{be}{(1-b)(1-e)} + \tfrac{cd}{(1-c)(1-d)} + \tfrac{ce}{(1-c)(1-e)} \\
&+ \tfrac{abc}{(1-a)(1-b)(1-c)} + \tfrac{abd}{(1-a)(1-b)(1-d)} + \tfrac{acd}{(1-a)(1-c)(1-d)} + \tfrac{bcd}{(1-b)(1-c)(1-d)} + \tfrac{bce}{(1-b)(1-c)(1-e)} \\
&= \frac{1 - abcd - abe - ace - de + abce + abde + acde}{(1-a)(1-b)(1-c)(1-d)(1-e)}.
\end{aligned}
$$

See Example 1.5, below, for a quick way to get this last equality. □

The formula for the Hilbert series of S/I_Δ perhaps becomes a little neater when we coarsen to the $\mathbb{Z}$-grading. Letting $f_i := |F_i(\Delta)|$ = the number of i-faces of Δ (Section 0.5), we get

$$
\begin{aligned}
H(S/I_\Delta; t, \ldots, t) &= \frac{1}{(1-t)^n} \sum_{i=0}^{d} f_{i-1}\, t^i (1-t)^{n-i} \\
&= \frac{h_0 + h_1 t + h_2 t^2 + \cdots + h_d t^d}{(1-t)^d}
\end{aligned}
$$

where $d = \dim(\Delta)+1$ and the vector $(h_0, h_1, \ldots, h_d)$, which is defined by this equation, is the *h-vector* of Δ. We will provide no more generalities about the h-vector or the f-vector $(f_{-1}, f_0, \ldots, f_d)$ since they are, to some approximation, the subjects of a whole chapter of Stanley's book [32].

1.3 Free resolutions

There is no general formula for the maps in a minimal free resolution of an arbitrary squarefree monomial ideal I_Δ. However, we can figure out what the Betti numbers are in terms of the simplicial cohomology of the Stanley-Reisner complex Δ. Before doing this, let's first have a look at some free resolutions. In what follows, it is convenient and customary to identify the subset $\sigma \subseteq \{1, \ldots, n\}$ with its characteristic vector in $\{0,1\}^n$, which has a 1 in the i^{th} slot when $i \in \sigma$. This causes the notation $\mathbf{x}^\sigma = \prod_{i\in\sigma} x_i$ to make even more sense than it did in Example 0.1.3.

Example 1.5 Let Γ be the simplicial complex from Example 1.4. The Stanley-Reisner ring S/I_Γ has minimal free resolution

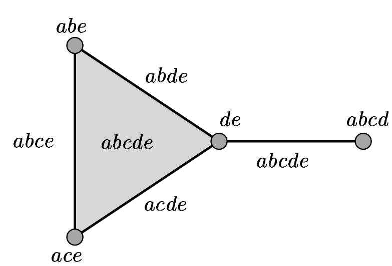
$$0 \longleftarrow S \xleftarrow{\begin{array}{c|cccc} & de & abe & ace & abcd \\ \hline 1 & 1 & 1 & 1 & 1 \end{array}} S^4 \xleftarrow{\begin{array}{c|cccc} & abce & abde & acde & abcde \\ \hline de & 0 & -1 & -1 & -1 \\ abe & 1 & 1 & 0 & 0 \\ ace & -1 & 0 & 1 & 0 \\ abcd & 0 & 0 & 0 & 1 \end{array}} S^4 \xleftarrow{\begin{array}{c|c} & abcde \\ \hline abce & -1 \\ abde & 1 \\ acde & -1 \\ abcde & 0 \end{array}} S \longleftarrow 0$$

where the maps are denoted by *monomial matrices* as in Section 0.2. For an example of how to recover the usual matrix notation for maps of free S-modules, the middle matrix can be written as

$$\begin{pmatrix} 0 & -ab & -ac & -abc \\ c & d & 0 & 0 \\ -b & 0 & d & 0 \\ 0 & 0 & 0 & e \end{pmatrix}$$

without the border entries.

As a preview to Lecture V, the reader is invited to figure out how the following labeled simplicial complex corresponds to the above free resolution:



Hint: compare the free resolution and the labeled simplicial complex with the numerator of the Hilbert series in Example 1.4. □

Example 1.6 The minimal free resolution of $k = S/\mathfrak{m}$ is the *Koszul complex* $\mathbb{K}_{\bullet}$, and is easiest to describe using monomial matrices. Recall that in the reduced chain complex of the simplex consisting of all subsets of $\{1, \ldots, n\}$, the basis vectors are called e_σ for $\sigma \subseteq \{1, \ldots, n\}$. The monomial matrices for the maps in $\mathbb{K}_{\bullet}$ are obtained simply by labeling the column and the row corresponding to e_σ by σ itself (or $\mathbf{x}^\sigma$) and shifting homologically so that the empty set $\varnothing$ is in homological degree 0. If $n = 3$, for instance, we get the following resolution (with x_1, x_2, x_3 replaced by x, y, z).

$$\mathbb{K}_{\bullet}: \quad 0 \longleftarrow S \xleftarrow{\begin{array}{c|ccc} & x & y & z \\ \hline 1 & 1 & 1 & 1 \end{array}} S^3 \xleftarrow{\begin{array}{c|ccc} & yz & xz & xy \\ \hline x & 0 & 1 & 1 \\ y & 1 & 0 & -1 \\ z & -1 & -1 & 0 \end{array}} S^3 \xleftarrow{\begin{array}{c|c} & xyz \\ \hline yz & 1 \\ xz & -1 \\ xy & 1 \end{array}} S \longleftarrow 0$$

The method of proof for many statements about monomial ideals is to determine what is happening in each $\mathbb{Z}^n$-graded degree of a complex of S-modules. To illustrate, we do this now for $\mathbb{K}_{\bullet}$ in some detail.

The essential observation is that a free module generated by 1_τ in degree τ is nonzero in degree σ precisely when $\tau \subseteq \sigma$ (equivalently, when $\mathbf{x}^\tau$ divides $\mathbf{x}^\sigma$). It helps to think of $S \cdot 1_\tau$ as the principal ideal $\langle \mathbf{x}^\tau \rangle$, so the statement becomes

$$\langle \mathbf{x}^\tau \rangle_\sigma \neq 0 \iff \mathbf{x}^\sigma \in \langle \mathbf{x}^\tau \rangle \iff \mathbf{x}^\tau \text{ divides } \mathbf{x}^\sigma .$$

The only contribution to the degree $\mathbf{0}$ part of $\mathbb{K}_{\bullet}$, for example, comes from the free module corresponding to $\varnothing$, which is generated by $1_\varnothing$ in degree $\mathbf{0}$. More generally, the degree σ part of $\mathbb{K}_{\bullet}$ comes from those rows and columns labeled by faces of σ. In other words, we restrict $\mathbb{K}_{\bullet}$ to degree σ by ignoring summands $S \cdot 1_\tau$ for which τ is not a face of σ. Therefore, $(\mathbb{K}_{\bullet})_\sigma$ is, as a chain complex of k-vector spaces, precisely equal to the reduced chain complex of the simplex σ! This explains why the homology of $\mathbb{K}_{\bullet}$ is just k in degree $\mathbf{0}$ and zero elsewhere: a simplex σ is contractible, so it has no reduced homology (unless $\sigma = \varnothing$—see Remark 0.5). □

Example 1.7 Instead of using the reduced chain complex of the simplex $\{1, \ldots, n\}$, we could use the reduced *cochain* complex to get another version of the Koszul complex. This time, we denote it by $\mathbb{K}^{\bullet}$ and put the label $\mathbf{x}^\sigma$ on the column and row corresponding to $e^*_{\overline{\sigma}}$, where $\overline{\sigma} = \{1, \ldots, n\} \setminus \sigma$. Of course, if we want to consider $\mathbb{K}^{\bullet}$ as a free resolution of k, we must shift the homological degrees so that $e^*_\varnothing$ sits in homological degree n, and more generally $e^*_{\overline{\sigma}}$ sits in homological degree $n - |\overline{\sigma}| = |\sigma|$.

As a complex of k-vector spaces, $(\mathbb{K}^{\bullet})_\sigma = \bigoplus_{\tau \subseteq \sigma} S[\tau - \sigma]_\sigma$ is the subcomplex of the cochain complex $\widetilde{\mathcal{C}}^{\bullet}(\{1, \ldots, n\}; k)$ spanned by the basis elements $\{e^*_{\overline{\tau}} \mid \tau \subseteq \sigma\} = \{e^*_{\tau \cup \overline{\sigma}} \mid \tau \subseteq \sigma\} = \{\mathbf{x}^\tau \cdot 1_{\sigma - \tau} \mid \tau \subseteq \sigma\}$. In fact, though,

$$\begin{array}{rcl} (\mathbb{K}^{\bullet})_\sigma & \cong & \widetilde{\mathcal{C}}^{\bullet}(\sigma) \\ e^*_{\tau \cup \overline{\sigma}} & \mapsto & \operatorname{sign}(\tau, \overline{\sigma}) e^*_\tau \end{array}$$

where $\operatorname{sign}(\tau, \overline{\sigma})$ is the sign of the permutation which puts the list $(\tau, \overline{\sigma})$ into increasing order. This point of view will be essential for the proof of Theorem 1.8. Just as in the last example, this explains why $\mathbb{K}^{\bullet}$ has homology k in degree $\mathbf{0}$ and 0 otherwise. □

For each $\sigma \subseteq \{1, \ldots, n\}$, define the *restriction of* Δ *to* σ by

$$\Delta|_\sigma := \{\tau \in \Delta \mid \tau \subseteq \sigma\}.$$

Hochster's formula expresses the number of summands generated in degree σ at the i^{th} stage in a minimal free resolution of S/I_Δ as follows.

Theorem 1.8 (Hochster [19]) $\beta_{i-1,\sigma}(I_\Delta) = \beta_{i,\sigma}(S/I_\Delta) = \dim_k \widetilde{H}^{|\sigma|-i-1}(\Delta|_\sigma; k)$.

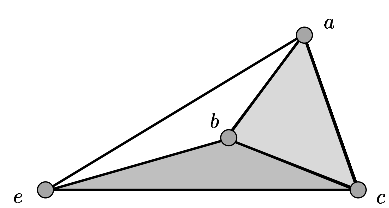
It turns out that all of the syzygies occur in degrees given by some incidence vector σ; we will have a quick and easy proof available as soon as we introduce the *hull resolution* in Lecture V (see the comment after Theorem 5.9). For now, we apply this fact to note that Hochster's formula accounts for all of the nonzero Betti numbers.

Proof: The first equality is obvious, since a minimal free resolution of I_Δ is achieved by snipping off the copy of S occurring in homological degree 0 of the minimal free resolution of S/I_Δ. For the second equality, we use the commutativity of Tor: we can calculate $\text{Tor}_i^S(k, S/I_\Delta)$ by tensoring the Koszul complex $\mathbb{K}^\bullet$ with S/I_Δ as follows.

In each squarefree degree, $(S/I_\Delta \otimes_S \mathbb{K}^\bullet)_\sigma$ is a quotient of $(\mathbb{K}^\bullet)_\sigma$ and, as in Example 1.7, will be the reduced cochain complex of some simplicial complex. For $\tau \subseteq \sigma$, the basis vector $e^*_{\tau\cup\overline{\sigma}} = \mathbf{x}^\tau \otimes 1_{\sigma-\tau} \in (\mathbb{K}^\bullet)_\sigma$ becomes zero in the tensor product $S/I_\Delta \otimes \mathbb{K}^\bullet$ if and only if $\mathbf{x}^\tau = 0$ in S/I_Δ, and this occurs if and only if $\tau \notin \Delta$. Therefore, the k-basis for $(S/I_\Delta \otimes \mathbb{K}^\bullet)_\sigma$ is $\{e^*_{\tau\cup\overline{\sigma}} \mid \tau \in \Delta|_\sigma\}$. Under the isomorphism in Example 1.7, we find that $(S/I_\Delta \otimes \mathbb{K}^\bullet)_\sigma \cong \widetilde{\mathcal{C}}^\bullet(\Delta|_\sigma; k)$, but considered as a homological complex (decreasing indices) with $e^*_\varnothing$ in homological degree $|\sigma|$, and more generally e^*_τ in homological degree $|\sigma| - |\tau| = |\sigma| - \dim \tau - 1$. Taking the i^{th} homology of this complex yields $\widetilde{H}^{|\sigma|-i-1}(\Delta|_\sigma; k)$, as desired. □

Example 1.9 Let Γ be as in Examples 1.4 and 1.5. Taking $\sigma = \{a, b, c, d, e\}$, corresponding to the monomial $abcde$, we have $\Gamma|_\sigma = \Gamma$. Therefore, we can use Hochster's formula to compute the dimensions of the cohomology groups of Γ. From the labelings of the matrices, we see $\beta_{3,\sigma}(S/I_\Gamma) = \beta_{2,\sigma}(S/I_\Gamma) = 1$, and the other Betti numbers in this degree are zero. Thus, $\widetilde{H}^1(\Gamma; k) \cong \widetilde{H}^2(\Gamma; k) \cong k$, and the other reduced cohomology groups of Γ are 0. The nonzero cohomology comes from the "empty" circle $\{a, b, e\}$ and the "empty" sphere $\{a, b, c, d\}$.

For another example, take $\sigma = \{a, b, c, e\}$, corresponding to the monomial $abce$. The restriction $\Gamma|_\sigma$ is the simplicial complex



Hochster's formula gives $\widetilde{H}^{|\sigma|-1-2}(\Gamma|_\sigma; k) = \widetilde{H}^1(\Gamma|_\sigma; k) \cong k$, and the other cohomology groups are trivial. □

Remark 1.10 Since we are working over a field k, the reader who wishes may substitute reduced homology for cohomology when calculating Betti numbers, since these have the same dimension. □

Further reading The standard reference for squarefree monomial ideals is Stanley [32], but Chapter 5 of the excellent book of Bruns and Herzog [7] is also recommended.

2 Lecture II: Borel-fixed monomial ideals

Squarefree monomial ideals occur mostly in a combinatorial context. Our next monomial ideals, the *Borel-fixed* monomial ideals, have more direct connection to algebraic geometry, where they arise as fixed points of an algebraic group action on the Hilbert scheme. But don't worry: one need not know what the Hilbert scheme is to understand both the group action and the fixed points.

2.1 Group actions

Let's begin by putting this group action in perspective. *Throughout this lecture, the field k has characteristic 0, and all ideals of S that we consider are $\mathbb{Z}$-graded.* We have the following inclusions of matrix groups:

$$\begin{array}{rcll} \mathrm{GL}_n(k) & = & \{\text{invertible } n\times n \text{ matrices}\} & \text{general linear group} \\ \cup & & & \\ B_n(k) & = & \{\text{upper triangular matrices}\} & \text{Borel group} \\ \cup & & & \\ T_n(k) & = & \{\text{diagonal matrices}\} & \text{torus group} \end{array}$$

The general linear group (and hence its subgroups, as well) acts on the polynomial ring as follows. For $g = (g_{ij}) \in \mathrm{GL}_n(k)$ and $f = f(x_1, \ldots, x_n) \in S$, let g act on f by

$$g \cdot f = f(gx_1, \ldots, gx_n) \quad \text{where} \quad gx_j := \sum_{i=1}^{n} g_{ij} x_i \, .$$

Given an ideal $I \subset S$, we can then set

$$g \cdot I = \{g \cdot f \mid f \in I\} \, ,$$

and this defines the action on the Hilbert scheme, whose points correspond to homogeneous ideals of S up to primary components at the irrelevant ideal $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$.

Being fixed by all of GL_n is awfully difficult for an ideal I; the only way this can happen is if $I = 0$ or I is some power $\mathfrak{m}^d$. At the other extreme, an ideal I is torus fixed if and only if I is a monomial ideal. This characterization of monomial ideals was one of the original motivations for studying *toric varieties*, examples of which come from the orbit-closures under T_n of ideals I. In any case, it explains why we are interested in the action of B_n, which will therefore pick out some special kinds of monomial ideals (but not so special as to be powers of $\mathfrak{m}$!). The extra property enjoyed by a Borel-fixed ideal is that smaller-indexed variables can be swapped in for larger ones without leaving the ideal (see, for instance, [12, Theorem 15.23]):

Proposition 2.1 *The following are equivalent for a monomial ideal I:*

1. *I is Borel-fixed;*
2. *If $m \in I$ is any monomial divisible by x_j, then $mx_i/x_j \in I$ for $i < j$.*

To test your understanding, try answering the following questions:

1. An irreducible ideal $I = \langle x_{i_1}^{a_1}, \ldots, x_{i_r}^{a_r} \rangle$ is Borel fixed if and only if ______?
2. Are the primary components of a Borel-fixed ideal Borel-fixed?

2.2 Generic initial ideals

Again let $I \subset S$ be a $\mathbb{Z}$-graded ideal, and let $<$ be any term order (for an introduction to term orders and the other material in this Section, see [12, Chapter 15]). Every $g \in \mathrm{GL}_n$ determines a monomial ideal $\mathrm{in}_<(g \cdot I)$, the *initial monomial ideal* of $g \cdot I$ for the term order $<$. It is a theorem that as a function of g, $\mathrm{in}_<(g \cdot I)$ is constant on a Zariski open subset of GL_n. The constant value on that open set is called the the *generic initial ideal* of I for the term order $<$, and is denoted

$$\mathrm{gin}_<(I) := \mathrm{in}_<(g \cdot I)\,.$$

The point of all this is:

Theorem 2.2 $\mathrm{gin}_<(I)$ *is Borel-fixed.*

See [12, Chapter 15] for a proof.

Example 2.3 Let $f, g \in k[x_1, x_2, x_3, x_4]$ be generic forms of degrees d, e, respectively. Here is the ideal $M = \mathrm{gin}_{\mathrm{lex}}(\langle f, g \rangle)$ for a few cases:

$$\begin{array}{lrcl}
(d,e) = (2,2) & M & = & \langle x_2^4, x_1x_3^2, x_1x_2, x_1^2 \rangle \\
 & & = & \langle x_1, x_2^4 \rangle \cap \langle x_1^2, x_2, x_3^2 \rangle \\
(d,e) = (2,3) & M & = & \langle x_2^6, x_1x_3^6, x_1x_2x_4^4, x_1x_2x_3x_4^2, x_1x_2x_3^2, x_1x_2^2, x_1^2 \rangle \\
 & & = & \langle x_1, x_2^6 \rangle \cap \langle x_1^2, x_2, x_3^6 \rangle \cap \langle x_1^2, x_2^2, x_3, x_4^4 \rangle \cap \langle x_1^2, x_2^2, x_3^2, x_4^2 \rangle \\
(d,e) = (3,3) & M & = & \langle x_2^9, x_1x_3^{18}, x_1x_2x_4^{16}, x_1x_2x_3x_4^{14}, \ldots, x_1^3 \rangle \;\; \text{(26 generators)}
\end{array}$$

2.3 The Eliahou-Kervaire resolution

Throughout this Section, let the monomials $m_1, \ldots, m_r$ minimally generate a Borel-fixed ideal, and let u_i be the largest index of a variable dividing m_i.

Lemma 2.4 *Any monomial $m \in \langle m_1, \ldots, m_r \rangle$ can be written uniquely as a product $m = m_i m'$ such that $u_i \leq$ the smallest index of an indeterminate dividing m'.*

Proof: Uniqueness: Suppose $m = m_i m'_i = m_j m'_j$ both satisfy the condition, with $u_i \leq u_j$. Then m_i and m_j agree in every variable whose index is $< u_i$. Now if x_{u_i} divides m'_j then $u_i = u_j$ by the assumed condition, whence one of m_i and m_j divides the other, so $i = j$. Otherwise, x_{u_i} does not divide m'_j. In this case the degree of x_{u_i} in m_i is $\leq$ the degree of x_{u_i} in m_j, which equals the degree of x_{u_i} in m, so that again m_i divides m_j and $i = j$.

Existence: Suppose that $m = m_j m'$ for some j, but that $u_j > u =$: the smallest index of a variable dividing m'. Then Proposition 2.1 says that we can replace m_j by any minimal generator m_i dividing $m_j x_u / x_{u_j}$. By construction, $u_i \leq u_j$, so either $u_i < u_j$, or $u_i = u_j$ and the degree of x_{u_i} in m_i is $\leq$ the degree of x_{u_i} in m_j. This shows that we can't keep going on making such replacements forever. □

Now apply this lemma to each $m = m_i x_u$ where $u < u_i$. If $m = m_i x_u = m_j m'$ as in the Lemma, then $x_u \mathbf{e}_i - m' \mathbf{e}_j$ is a syzygy of the ideal. In this way, we get a set of minimal first syzygies for a Borel-fixed ideal.

Example 2.5 A Borel-fixed ideal and its minimal first syzygies:

$$
\begin{array}{ccccccc}
\langle x_1x_2x_4^4, & x_1x_2x_3x_4^2, & x_1x_3^6, & x_1x_2x_3^2, & x_2^6, & x_1x_2^2, & x_1^2 \rangle \\
x_3\,\mathbf{e}_1 & -x_4^2\,\mathbf{e}_2 & & & & & \\
x_2\,\mathbf{e}_1 & & & & & -x_4^4\,\mathbf{e}_6 & \\
x_1\,\mathbf{e}_1 & & & & & & -x_2x_4^4\,\mathbf{e}_7 \\
 & x_3\,\mathbf{e}_2 & & -x_4^2\,\mathbf{e}_4 & & & \\
 & x_2\,\mathbf{e}_2 & & & & -x_3x_4^2\,\mathbf{e}_6 & \\
 & x_1\,\mathbf{e}_2 & & & & & -x_2x_3x_4^2\,\mathbf{e}_7 \\
 & & x_2\,\mathbf{e}_3 & -x_3^4\,\mathbf{e}_4 & & & \\
 & & x_1\,\mathbf{e}_3 & & & & -x_3^6\,\mathbf{e}_7 \\
 & & & x_2\,\mathbf{e}_4 & & -x_3^2\,\mathbf{e}_6 & \\
 & & & x_1\,\mathbf{e}_4 & & & -x_2x_3^2\,\mathbf{e}_7 \\
 & & & & x_1\,\mathbf{e}_5 & -x_2^4\,\mathbf{e}_6 & \\
 & & & & & x_1\,\mathbf{e}_6 & -x_2^2\,\mathbf{e}_7
\end{array}
$$

Key Fact: These syzygies form the reduced Gröbner basis for the presentation module $\mathcal{M}$ of the Borel-fixed ideal with respect to any position-over-term (POT) order. Indeed, the initial terms of the syzygies above are relatively prime by [10, Proposition 2.9.4]. In the specific example here, the initial module is

$$
\begin{array}{rllll}
\mathrm{in}(\mathcal{M}) = & \langle x_1\,\mathbf{e}_1, & x_2\,\mathbf{e}_1, & x_3\,\mathbf{e}_1, & \\
 & x_1\,\mathbf{e}_2, & x_2\,\mathbf{e}_2, & x_3\,\mathbf{e}_2, & \\
 & x_1\,\mathbf{e}_3, & x_2\,\mathbf{e}_3, & & \\
 & x_1\,\mathbf{e}_4, & x_2\,\mathbf{e}_4, & & \\
 & x_1\,\mathbf{e}_5, & & & \\
 & x_1\,\mathbf{e}_6 \rangle & & & \subset k[\mathbf{x}]^7
\end{array}
$$

Its resolution is a direct sum of Koszul complexes:

$$
\begin{array}{ccccccccccc}
 & & S\,\mathbf{e}_1 & \longleftarrow & S^3 & \longleftarrow & S^3 & \longleftarrow & S & \longleftarrow & 0 & \\
 & & S\,\mathbf{e}_2 & \longleftarrow & S^3 & \longleftarrow & S^3 & \longleftarrow & S & \longleftarrow & 0 & \oplus \\
 & & S\,\mathbf{e}_3 & \longleftarrow & S^2 & \longleftarrow & S & \longleftarrow & 0 & & & \oplus \\
 & & S\,\mathbf{e}_4 & \longleftarrow & S^2 & \longleftarrow & S & \longleftarrow & 0 & & & \oplus \\
 & & S\,\mathbf{e}_5 & \longleftarrow & S & \longleftarrow & 0 & & & & & \oplus \\
 & & S\,\mathbf{e}_6 & \longleftarrow & S & \longleftarrow & 0 & & & & & \oplus \\
 & & & & & & & & & & & \\
0 & \longleftarrow & \mathrm{in}\,\mathcal{M} & \longleftarrow & S^{12} & \longleftarrow & S^8 & \longleftarrow & S^2 & \longleftarrow & 0 &
\end{array}
$$

The resolution of $\mathrm{in}(\mathcal{M})$ is linear and lifts (by adding trailing terms) to the minimal *Eliahou-Kervaire* resolution of $\mathcal{M} \subset S^7$ and of $S^7/\mathcal{M}$, the given Borel-fixed ideal [14].

$$
0 \longleftarrow S \xleftarrow{(x_1x_2x_4^4 \quad x_1x_2x_3x_4^2 \quad \cdots \quad x_1^2)} S^7 \longleftarrow S^{12} \longleftarrow S^8 \longleftarrow S^2 \longleftarrow 0
$$

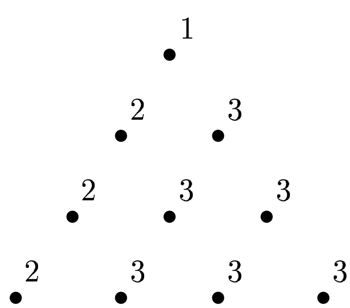
This lifting of linear resolutions of initial modules is a general phenomenon, being a consequence of the upper-semicontinuity of Betti numbers in flat families. And for Borel-fixed ideals, the Koszul behavior exhibited above in the resolution of the initial module is fundamental. Therefore, because we understand the numerics of Koszul complexes, we understand the numerics of Borel-fixed ideals:

Theorem 2.6 (Eliahou-Kervaire [14]) *If $I = \langle m_1, \ldots, m_r \rangle$ is Borel-fixed and u_i is the largest index of a variable dividing m_i, then*

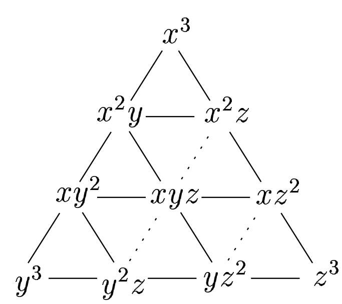
1. *The number of j-th syzygies of I is $\sum_{i=1}^{r} \binom{u_i - 1}{j}$, and*
2. *The numerator of the Hilbert series is I equals*

$$
1 - \sum_{i=1}^{r} m_i \prod_{j=1}^{u_i - 1} (1 - x_j)\,.
$$

Example 2.7 Powers $\mathfrak{m}^d$ of the maximal ideal are Borel-fixed and hence resolved by Eliahou-Kervaire. In the case $n = d = 3$, we get:



u_i



$\langle x, y, z \rangle^3$

$$S \longleftarrow S^{10} \longleftarrow S^{15} \longleftarrow S^{6} \longleftarrow 0$$

$$\begin{array}{ccc} \begin{array}{c} 1 \\ 1\;1 \\ 1\;1\;1 \\ 1\;1\;1\;1 \end{array} & \begin{array}{c} 0 \\ 1\;2 \\ 1\;2\;2 \\ 1\;2\;2\;2 \end{array} & \begin{array}{c} 0 \\ 0\;1 \\ 0\;1\;1 \\ 0\;1\;1\;1 \end{array} \end{array}$$

The triangles of numbers below the resolution are the $\binom{u_i-1}{j}$ of Theorem 2.6; they indicate how the initial module decomposes as a direct sum of Koszul complexes. It may seem that any minimal resolution breaks the symmetry under the group S_3 permuting the variables, but this is not necessarily so—see [25]. □

2.4 Lex-segment ideals

Let $<$ be a term order and $H\colon \mathbb{N} \to \mathbb{N}$ the $\mathbb{Z}$-graded Hilbert function of a homogeneous ideal I. The *segment ideal*, $I_{H,<}$, is the ideal k-spanned by the first $H(i)$ monomials with respect to $<$ in each degree i. It is Borel-fixed. If $<$ is the lexicographical order, we get the *lex-segment* ideal, $I_{H,\text{lex}}$. Historically, the reason for studying lex-segment ideals was because of their supremely bad (good?) numerical behavior.

Theorem 2.8 (Macaulay, [22]) *$I_{H,\text{lex}}$ has the highest degree generators among all (monomial) ideals with the same coarse Hilbert function.*

In Macaulay's theorem, it is enough to restrict our attention to monomial ideals, since any initial ideal of a $\mathbb{Z}$-graded ideal I has generators with at least the degrees of the generators of I.

The degrees of the generators, of course, are measuring the zeroth Betti numbers. One can also ask which ideals have the worst behavior with respect to the degrees of the higher Betti numbers. The ultimate statement is that lex-segment ideals take the cake, simultaneously for all Betti numbers.

Theorem 2.9 (Bigatti, Hulett [5, 21]) *$I_{H,\text{lex}}$ has the most minimal i-th syzygies for all i, among all (monomial) ideals with the same Hilbert function.*

Again, the upper semi-continuity of Betti numbers implies that we need only compare the lex-segment ideals with other monomial ideals.

3 Lecture III: Monomial ideals in three variables

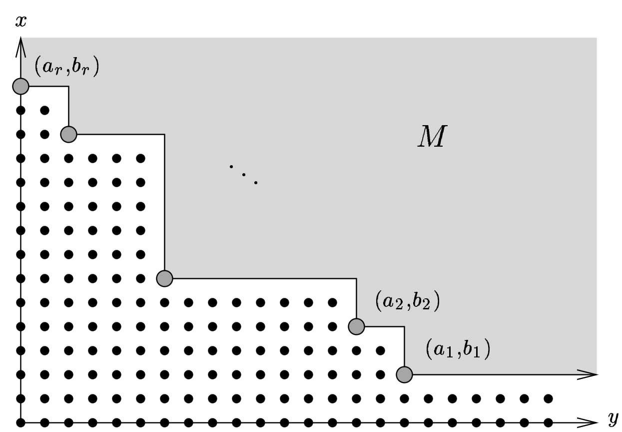
Squarefree and Borel-fixed ideals each have their own advantages, the former yielding insight into combinatorics, and the latter into extremal numerical behavior in algebraic geometry. Their utility stems in both cases from our ability to express the appropriate information in terms of the defining properties of these special classes of monomial ideals; and in the Borel-fixed case, we are actually able to write down an explicit minimal free resolution.

However, this is not possible for general monomial ideals, at least not without making arbitrary choices. Even in the Borel-fixed case, the choices have really already been made for us—in the order of the variables, for instance—and it may well be that an ideal is Borel-fixed with respect to more than one such order (e.g. the powers of the irrelevant ideal $\mathfrak{m}$). This inability to write down explicit canonical minimal (or at least "small") resolutions has prompted research into the intrinsic *geometric* properties of monomial ideals, resulting from the inclusion of $\mathbb{Z}^n$ into $\mathbb{R}^n$. Consequently, convex geometric techniques, along with the combinatorial and algebraic topological methods surrounding them, are now being used to derive ways of expressing information for general monomial ideals which were until now available only for special classes.

The purpose of this lecture is to give a heuristic introduction to these geometric ideas, in the case of two and three variables. The details of some of the multiple facets of this theory in higher dimensions are the subjects of the remaining five lectures. Many parts of the exposition in this lecture have been adapted from the introductory article of Miller and Sturmfels [25].

3.1 Monomial ideals in two variables

Let $M = \langle m_1, \ldots, m_r \rangle = \langle x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, \ldots, x^{a_r}y^{b_r} \rangle$ be a monomial ideal with $a_1 > a_2 > \cdots > a_r$ and $b_1 < b_2 < \cdots < b_r$. The *staircase diagram* for M shows the boundary between the region of the plane containing the (exponent vectors of) monomials in M and those not in M:



The black lattice points, contained completely within the non-shaded region, form a k-basis for S/M. As we have already seen in Section 0.4, the Hilbert series $H(S/M;\mathbf{x})$ is therefore the sum of all monomials not in M. But for practical purposes, we need to have $H(S/M;\mathbf{x})$ written as a rational function.

One way to accomplish this is by inclusion-exclusion. Start with all of the monomials in S. Then, for each minimal generator m_i, subtract off the monomials in the principal ideal $\langle m_i \rangle$ (which looks like a shifted positive orthant). Of course, now we've subtracted the monomials in $\langle m_i \rangle \cap \langle m_j \rangle = \langle \mathrm{lcm}\,(m_i, m_j) \rangle$ too many times, so we have to add those back in. Continuing in this way, we eventually (after at most r steps) have counted each monomial the right number of times. But this procedure produces *way* more terms than are necessary; almost all of them cancel, in the end.

There is a more efficient way, though, to do the inclusion-exclusion: after we've added in the principal ideals $\langle m_i \rangle$, we subtract of not *all* of the principal ideals $\langle \mathrm{lcm}\,(m_i, m_j) \rangle$, but only those which come from *adjacent* m_i and m_j. This yields the Hilbert series after just a couple of steps. We find that the numerator of the Hilbert series is

$$\begin{aligned}
(1-x)(1-y)\,H(S/M;x,y) &= (1-x)(1-y) \sum_{x^i y^j \notin M} x^i y^j \\
\text{(by inclusion exclusion)} &= \sum_{I \subseteq \{1,\dots,r\}} (-1)^I \,\mathrm{lcm}(x^{a_i} y^{b_i} \mid i \in I) \\
\text{(more efficient inclusion exclusion)} &= 1 - \sum_{i=1}^{r} x^{a_i} y^{b_i} + \sum_{j=1}^{r-1} x^{a_j} y^{b_{j+1}} \\
&= 1 - \text{inner corners} + \text{outer corners}.
\end{aligned}$$

The inclusion-exclusion process is in fact making a highly non-minimal free resolution of S/M called the *Taylor resolution* (Section 5.3). Our more efficient way of doing things in fact yields the *minimal* free resolution

$$0 \longleftarrow S \longleftarrow S^r \longleftarrow S^{r-1} \longleftarrow 0\,.$$

The important point is that the adjacent pairs of generators are the minimal first syzygies: $y^{b_{i+1}-b_i}\mathbf{e}_i - x^{a_i - a_{i+1}}\mathbf{e}_{i+1}$.

3.2 Buchberger's second criterion

Finding minimal sets of syzygies for monomial ideals has an impact on algorithmic computation for arbitrary ideals. The connection is, of course, through Gröbner bases. Recall that a set of polynomials

$$f_i := m_i + \text{trailing terms under term order} <, \quad i = 1, 2, \dots, r$$

is a *Gröbner basis* under the term order $<$ if each S-pair

$$S(f_i, f_j) := \frac{\mathrm{lcm}\,(m_i, m_j)}{m_i} f_i - \frac{\mathrm{lcm}\,(m_i, m_j)}{m_j} f_j$$

can be reduced to zero by $\{f_1, \dots, f_r\}$ using the *division algorithm*. *Buchberger's second criterion* says that it's actually enough to check this reduction to zero only for the S-pairs $S(f_i, f_j)$ corresponding to a generating set for the first syzygies on $M = \langle m_1, \dots, m_r \rangle$,

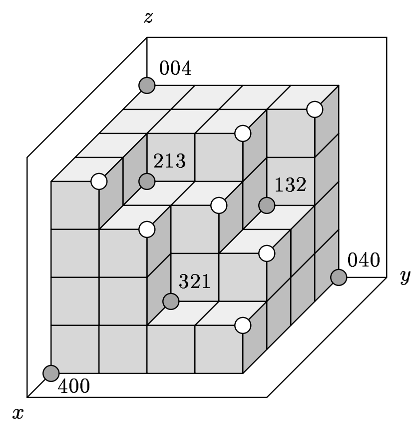
$$\left\{ \frac{\mathrm{lcm}\,(m_i, m_j)}{m_i}\mathbf{e}_i - \frac{\mathrm{lcm}(m_i, m_j)}{m_j}\mathbf{e}_j \right\}.$$

Thus, we can ignore those S-pairs $S(f_i, f_j)$ such that $\mathrm{lcm}\,(m_i, m_j)$ is a multiple of m_ℓ for $i < \ell < j$ or a proper multiple of other m_k (see [15]). For example, in the two-variable case, we need only consider the $r-1$ consecutive S-pairs instead of all $\binom{r}{2}$ pairs.

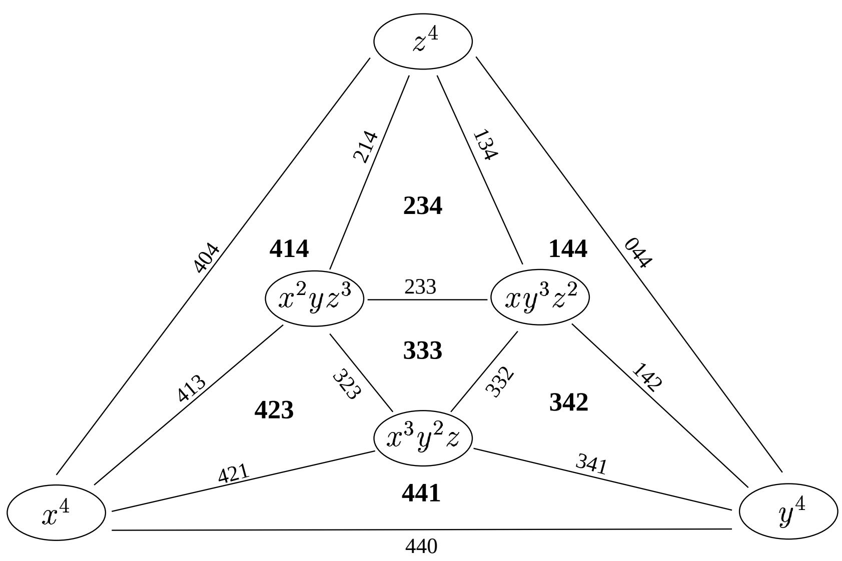
One of the original motivations for the definition of *generic* monomial ideals in Lecture IV is the way in which Buchberger's second criterion becomes simplified in the presence of certain randomness properties for generators of monomial ideals: genericity implies that whenever $\operatorname{lcm}(m_i, m_j)$ is a multiple of m_ℓ, it is automatically a proper multiple.

3.3 Resolution "by picture"

Staircase diagrams are also possible to draw for monomial ideals in three variables. For instance,



is a staircase diagram for the monomial ideal $M = \langle x^3y^2z, xy^3z^2, x^2yz^3, x^4, y^4, z^4\rangle$. Remember that the surface we see is the interface between being in or not in M, and that the lattice points strictly behind the interface are the ones not in M. Thus any lattice point which is visible in the staircase diagram is the exponent vector on a monomial in M. In particular, the dark dots correspond to the minimal generators of M—note how they sit in the "inner" corners.



Consider the graph above, in which we have connected the generators of M according to Buchberger's second criterion. Each edge and each triangular face is labeled by the exponent vector of the least common multiple of its vertices. As will be explained in the following lectures, much of the structure of the monomial ideal can be read off from this picture. For example, vertices correspond to generators, edges to first syzygies, and facets to second syzygies. In this particular case, where the monomial ideal is *artinian*, the facets also tell us the irreducible components, which correspond to the white dots on the "outer" corners of the staircase diagram. Overall, the kinds of information we can get include:

Irreducible decomposition (labels on triangles):

$$\begin{aligned} M &:= \langle x^4, y^4, z^4, x^3y^2z, xy^3z^2, x^2yz^3\rangle \\ &= \langle x^4, y^4, z\rangle \cap \langle x^4, y, z^4\rangle \cap \langle x, y^4, z^4\rangle \cap \langle x^4, y^2, z^3\rangle \cap \\ &\quad \langle x^3, y^4, z^2\rangle \cap \langle x^2, y^3, z^4\rangle \cap \langle x^3, y^3, z^3\rangle\,. \end{aligned}$$

Minimal free resolution (boundary complex of triangulation):

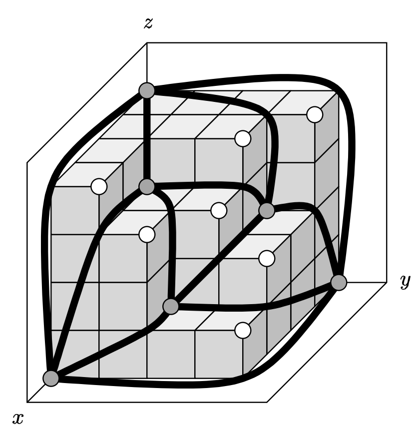
$$0 \leftarrow S \leftarrow S^6 \leftarrow S^{12} \leftarrow S^7 \leftarrow 0\,.$$

Numerator of the Hilbert series (alternating sum of all face labels):

$$1 - x^4 - \ldots - x^2yz^3 + x^4y^4 + \ldots + xy^3z^4 - x^4y^4z - \ldots - x^3y^3z^3\,.$$

3.4 Planar graphs

The graph produced in the previous example by Buchberger's second criterion can be embedded nicely into the staircase diagram: each edge consists of two straight segments connecting the minimal generators m_1 and m_2 to the exponent vector on $\mathrm{lcm}\,(m_1, m_2)$. It looks a little better if we replace the two straight segments by a spline going through m_1, $\mathrm{lcm}\,(m_1, m_2)$, and m_2:



Considering the graph as a subset of the 2-dimensional interface between M and $\{\text{not } M\}$, each region contains precisely one white dot situated on an outside corner, each vertex is a dark dot on inside corner, and each edge passes through one corner which is neither inside nor outside. The label on each vertex, edge, and region is the vector represented by the corresponding corner.

The monomial ideal above is special: it is *generic* (Lecture IV). This is why we didn't have any choices for where to put the edges—Buchberger's second criterion was enough. It turns out that a similar process works more generally, although we no longer get uniqueness (see [25] for exposition and [27] for details of the proof):

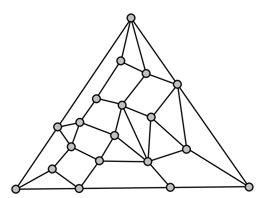
Theorem 3.1 *Every monomial ideal M in $k[x,y,z]$ has a minimal resolution by the bounded regions of a planar graph. That resolution gives irredundant formulas for the numerator of the Hilbert series and the irreducible decomposition of M.*

The vertices, edges, and bounded regions of this planar graph are labeled by their associated corners as in the example above. The free resolution is created by a method to be described precisely in Lecture V.

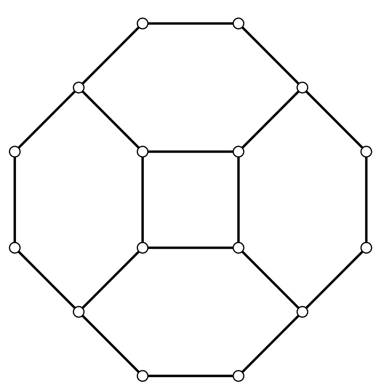
In general, since a planar graph with r vertices has at most $3r-6$ edges and $2r-5$ bounded regions, we get the following complexity result.

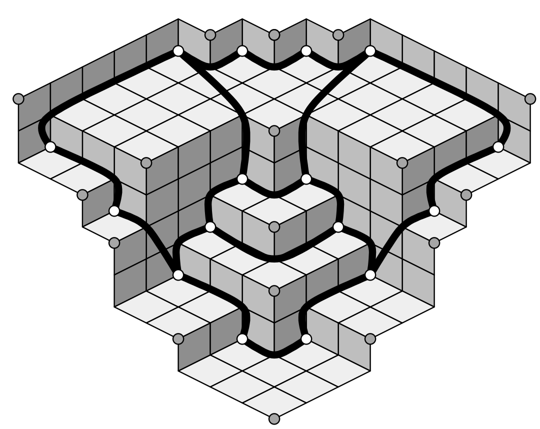
Corollary 3.2 *An ideal generated by r monomials in $k[x,y,z]$ has at most $3r-6$ minimal first syzygies and $2r-5$ minimal second syzygies.*

Conversely to Theorem 3.1, every 3-connected planar graph is the minimial free resolution of some monomial ideal in three variables [25]. For example, the reader is invited to find 19 monomials corresponding to the graph below:



Example 3.3 For another example, the monomial ideal below was constructed to have the given graph as its minimal free resolution:





The graph has been drawn in the staircase diagram in the manner described above. Notice that the order 8 symmetry of the graph is reduced to order 2 in the monomial ideal. Also, we haven't drawn in the coordinate axes; but the minimal free resolution only depends on the relative positions of the generators, not the absolute coordinates. On the other hand, the absolute positions will matter a great deal in Lecture VI. □

3.5 Reducing to the squarefree or Borel-fixed case

In the past, a standard way of treating homological and enumerative questions about arbitrary monomial ideals was to reduce to the cases we have discussed in previous lectures: squarefree or Borel-fixed ideals. The ideal

$$M = \langle x^4, y^4, z^4, x^3y^2z, xy^3z^2, x^2yz^3\rangle$$

can be reduced to a squarefree monomial ideal through *polarization*, wherein each power x^d of a variable is replaced by a product of d new variables:

$$\begin{aligned} M \rightsquigarrow I_\Delta \;\;&=\;\; \langle x_1x_2x_3x_4,\; y_1y_2y_3y_4,\; z_1z_2z_3z_4,\; x_1x_2x_3y_1y_2z_1, \\ &\quad\; x_1y_1y_2y_3z_1z_2,\; x_1x_2y_1z_1z_2z_3\rangle\,. \end{aligned}$$

However, polarization can make things much more complicated—the 8-dimensional simplicial complex Δ has 12 vertices and 51 facets and is hard compared to M itself.

On the other hand, we can reduce the same ideal M to a Borel-fixed ideal through *algebraic shifting:*

$$\begin{aligned} \mathrm{gin}_{\mathrm{revlex}}(M) \;\;&=\;\; \langle x^4, x^3y, x^2y^2, xy^4, y^5, x^3z^3, x^2yz^3, xy^3z^2, xy^2z^3, y^4z^2, \\ &\quad\; x^2z^5, xyz^5, xz^6, y^3z^4, y^2z^5, yz^6, z^7\rangle\,. \end{aligned}$$

Both ideals have colength 51, but the generic initial ideal is much more complicated than M itself, and the $\mathbb{N}^3$ grading is lost.

Compare both of the standard methods of reducing M to "easier" monomial ideals with the graph for M produced by Buchberger's second criterion presented earlier in the lecture. The remaining lectures are devoted to developing the latter point of view.

4 Lecture IV: Generic monomial ideals

We have already seen in Lecture II that monomial ideals derived from certain kinds of randomness have more concrete homological algebra. In this lecture we show how randomness of the exponent vectors on the minimal generators of a monomial ideal has similar consequences.

Before giving the definition, let us recall that the *support* of a monomial $m = \mathbf{x}^{\mathbf{a}}$ is the set $\mathrm{supp}(m) = \{i \in \{1,\dots,n\} \mid a_i \neq 0\}$, and $\mathbf{x}^{\mathrm{supp}(m)} = \prod_{i\in \mathrm{supp}(m)} x_i$. We say that a monomial m' *strictly divides* m if m' divides $m/\mathbf{x}^{\mathrm{supp}(m)}$.

Definition 4.1 A monomial ideal $M = \langle m_1,\dots,m_r\rangle$ is called *generic* if, whenever two distinct minimal generators m_i and m_j have the same positive degree in some variable x_s, there is a third generator m_ℓ which strictly divides $\mathrm{lcm}\,(m_i, m_j)$. M is called *strongly generic* if no two generators m_i and m_j have the same nonzero degree in any x_ℓ.

For example, $\langle x^2, xy, y^2z, z^2\rangle$ is strongly generic, $\langle x^2z, xy, y^2z, z^2\rangle$ is generic but not strongly generic, and $\langle x^2, xy, yz, z^2\rangle$ is not generic.

4.1 The Scarf complex

In the Preface, we promised to pick out finitely many monomials in a generic monomial ideal M which minimally determine the free resolution. We can actually define this finite set without assuming M is generic, although the set won't generally be as useful.

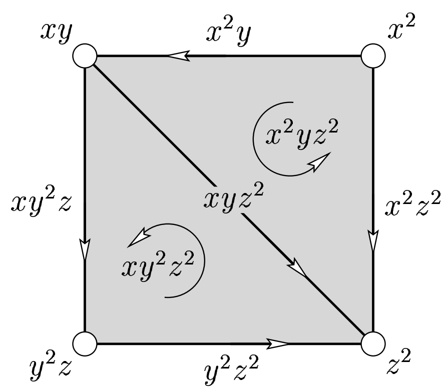
Definition 4.2 For $I \subseteq \{1, \ldots, r\}$, call the monomial $m_I = \operatorname{lcm}(m_i, i \in I)$ the *label* on I. The *Scarf complex* of M consists of sets of minimal generators with unique labels:

$$\Delta_M := \{I \subseteq \{1, \ldots, r\} \mid m_I = m_J \Longrightarrow I = J\}.$$

Lemma 4.3 *The Scarf complex Δ_M is a simplicial complex of dimension at most $n-1$.*

Proof: For each monomial m that can be expressed as a least common multiple of minimal generators of M, there is a unique maximal $I \subseteq \{1, \ldots, r\}$ such that $m = m_I$; this I consists of the indices of generators dividing m. In particular, if $I \in \Delta_M$ and m_J divides m_I, then $J \subseteq I$. Suppose that this is the case, and that J is the maximal subset of $\{1, \ldots, r\}$ with label m_J. To show that Δ_M is a simplicial complex, we show that if $j \in J$, then $m_{J \setminus j} \neq m_J$. But this holds because $m_{J\setminus j} = m_J \Rightarrow m_{I \setminus j} = m_I$. A facet I of Δ_M has cardinality at most n because for each index $i \in I$, the generator m_i contributes at least one coordinate to m_I—that is, there is some variable x_s such that m_i is the only generator dividing m_I and having the same degree in x_s as m_I. □

Example 4.4 Let $M = \langle x^2, xy, y^2z, z^2\rangle$. The Scarf complex of M is shown below, with each face accompanied by its monomial label.



Now we want to see how the minimal free resolution of S/M is obtained from the Scarf complex for generic M. Using monomial matrices, the construction is easy:

Definition 4.5 The *algebraic Scarf complex* $\mathbb{F}_{\Delta_M}$ is obtained by putting the reduced chain complex of Δ_M into a sequence of monomial matrices with the face label m_I on the row and column corresponding to $I \in \Delta_M$.

One need not start with a generic M in order for this monomial matrix to define a complex of free S-modules.

It is not hard to describe $\mathbb{F}_{\Delta_M}$ in more familiar terms, without referring to monomial matrices. Introduce a basis vector $\mathbf{e}_I$ in $\mathbb{Z}^n$-graded degree $\deg m_I$ and homological degree $|I|$ for each face I of Δ_M. Form the free S-module

$$\mathbb{F}_{\Delta_M} := \bigoplus_{I \in \Delta_M} S \cdot \mathbf{e}_I$$

with differential

$$\partial(\mathbf{e}_I) := \sum_{i \in I} \operatorname{sign}(i, I) \frac{m_I}{m_{I \setminus i}} \, \mathbf{e}_I.$$

Here, as in Section 0.5, $\operatorname{sign}(i, I) = (-1)^{j-1}$ if i is the j-th element of I when the elements of I are listed in increasing order.

Example 4.6 Let $M = \langle x^2, xy, y^2z, z^2 \rangle$ be as in Example 4.4. The algebraic Scarf complex $\mathbb{F}_{\Delta_M}$ is given by the sequence of monomial matrices below.

$$0 \longleftarrow S \xleftarrow{\begin{array}{c|cccc} & z^2 & y^2z & xy & x^2 \\ \hline 1 & 1 & 1 & 1 & 1 \end{array}} S^4 \xleftarrow{\begin{array}{c|ccccc} & x^2z^2 & x^2y & xy^2z & y^2z^2 & xyz^2 \\ \hline z^2 & 1 & 0 & 0 & 1 & 1 \\ y^2z & 0 & 0 & 1 & -1 & 0 \\ xy & 0 & 1 & -1 & 0 & -1 \\ x^2 & -1 & -1 & 0 & 0 & 0 \end{array}} S^5 \xleftarrow{\begin{array}{c|cc} & x^2yz^2 & xy^2z^2 \\ \hline x^2z^2 & -1 & 0 \\ x^2y & 1 & 0 \\ xy^2z & 0 & 1 \\ y^2z^2 & 0 & 1 \\ xyz^2 & 1 & -1 \end{array}} S^4 \longleftarrow 0$$

For an example of the non-monomial matrix way to write things,

$$\partial(\mathbf{e}_{234}) = z\mathbf{e}_{23} + x\mathbf{e}_{34} - y\mathbf{e}_{24} \,,$$

where $\mathbf{e}_{234}$ is the basis vector in degree xy^2z^2 corresponding to $\{2,3,4\} \subset \{1,2,3,4\}$.□

The theorem to which we have been building represents the kernel out of which grew most of the ideas in the rest of these lectures. It was introduced and proved by Bayer, Peeva, and Sturmfels [2] for strongly generic monomial ideals. Later, it was extended to generic ideals by Miller, Sturmfels, and Yanagawa [26]. We avoid giving the proof here, although parts of it follow from more general theorems on the *hull resolution* (whose proofs we also avoid) in Section 5.4.

Theorem 4.7 *The algebraic Scarf complex* $(\mathbb{F}_{\Delta_M}, \partial)$ *is contained in the minimal free resolution of* S/M*. For* M *generic,* $(\mathbb{F}_{\Delta_M}, \partial)$ *is a minimal free resolution of* S/M*.*

Recall that the *Euler characteristic* of Δ_M is the alternating sum $\sum_d (-1)^d f_d$ of the numbers of faces of varying dimensions. If we keep track of the monomial labels on the faces, then we obtain the numerator of the Hilbert series in terms of the labels on the Scarf complex.

Corollary 4.8 *The Hilbert series of S/M for a generic monomial ideal M is*

$$\frac{1}{1-x_1}\cdots\frac{1}{1-x_n}\cdot\sum_{I\in\Delta_M}(-1)^{|I|}m_I\,.$$

The numerator is the negative of the $\mathbb{Z}^n$-graded Euler characteristic *of* Δ_M.

Example 4.9 The numerator of the Hilbert series of the quotient S/M is

$$1-x^2-xy-y^2z-z^2+x^2z^2+x^2y+xy^2z+y^2z^2+xyz^2-x^2yz^2-xy^2z^2,$$

if $M=\langle x^2,xy,y^2z,z^2\rangle$ is the monomial ideal from the previous two examples. □

4.2 Deformation of exponents

In the previous lecture, we saw how questions about arbitrary monomial ideals could be reduced to questions about squarefree monomials ideals or about Borel-fixed ideals. We now introduce a process which transforms arbitrary monomial ideals to generic monomial ideals. If $M=\langle m_1,\ldots,m_r\rangle=\langle \mathbf{x}^{\mathbf{a}_1},\ldots,\mathbf{x}^{\mathbf{a}_r}\rangle$ is not generic, we choose a "nearby" generic ideal. As with genericity, the concept of deformation was originally introduced by Bayer, Peeva, and Sturmfels [2] but reworked by Miller, Sturmfels, and Yanagawa [26] to be more natural (genericity can be characterized in terms of invariance under deformation). Basically, one wants to add small real vectors to the exponent vectors on the generators of M without reversing any strict inequalities between the corresponding coordinates of any two generators. The point is to turn equalities into strict inequalities which can potentially go either way.

Definition 4.10 A *deformation* ϵ of a monomial ideal $M=\langle m_1,\ldots,m_r\rangle\subset S$ is a choice of vectors $\epsilon_i=(\epsilon_1^i,\ldots,\epsilon_n^i)\in\mathbb{R}^n$ for each $i\in\{1,\ldots,r\}$ satisfying

$$a_s^i<a_s^j \quad\Rightarrow\quad a_s^i+\epsilon_s^i<a_s^j+\epsilon_s^j \qquad\text{and}\qquad a_s^i=0\quad\Rightarrow\quad \epsilon_s^i=0\,,$$

where $\mathbf{a}_i=(a_1^i,\ldots,a_n^i)$ is the exponent vector of m_i. We formally introduce the monomial ideal (in a polynomial ring with real exponents):

$$M_\epsilon \quad := \quad \langle m_1\cdot\mathbf{x}^{\epsilon_1},m_2\cdot\mathbf{x}^{\epsilon_2},\ldots,m_r\cdot\mathbf{x}^{\epsilon_r}\rangle=\langle\mathbf{x}^{\mathbf{a}_1+\epsilon_1},\mathbf{x}^{\mathbf{a}_2+\epsilon_2},\ldots,\mathbf{x}^{\mathbf{a}_r+\epsilon_r}\rangle.$$

A deformation ϵ is called *generic* if M_ϵ is a generic monomial ideal.

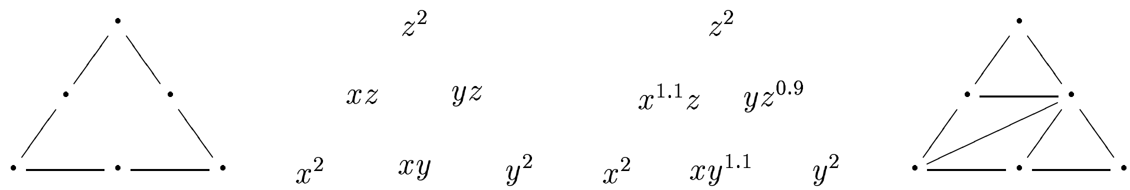
The Scarf complex Δ_{M_ϵ} of the deformation M_ϵ still makes sense, as a combinatorial object, and has the same vertex set $\{1,\ldots,r\}$ as Δ_M. The reader uncomfortable with real exponents can safely ignore them, since every combinatorial type of deformation can be obtained using only integers.

For generic ϵ, Δ_{M_ϵ} gives a simple (but typically non-minimal) free resolution of M. What we do is form the Scarf complex Δ_{M_ϵ} but label each face $I\in\Delta_{M_\epsilon}$ by m_I, not $\operatorname{lcm}(m_i\mathbf{x}^{\epsilon_i},i\in I)$.

Theorem 4.11 *The resulting complex $\mathbb{F}^{\epsilon}_{\Delta_M}$ of free S-modules is a resolution of S/M.*

By Lemma 4.3 this resolution has length less than or equal to the bound n provided by the Hilbert syzygy theorem, but is generally not minimal. Note that, unlike the reductions to squarefree or Borel-fixed ideals, this reduction to the generic situation actually produces a free resolution of S/M for any M. (Sticklers may argue that depolarization of a minimal free resolution of the polarization yields a resolution of the depolarization, but that's reducing the problem to one we also can't solve: finding the minimal free resolution of a squarefree monomial ideal.)

Example 4.12 The square $\mathfrak{m}^2$ of the irrelevant ideal is not generic, but we can find a generic deformation as depicted below. The resolution of $S/\mathfrak{m}^2$ afforded by the last diagram (with labels as in the second diagram) is not minimal.



Δ_M $\quad M \quad$ M_ϵ $\quad \Delta_{M_\epsilon}$

Note that the Scarf complex Δ_M is 1-dimensional, while Δ_{M_ϵ} is 2-dimensional. □

Seeing as how the Hilbert function is so easy to determine from a resolution, and how the resolution of a generic ideal is so easy to find, it is possible that generic deformation provides a useful algorithm for computing Hilbert series.

4.3 Triangulating the simplex

The Scarf complex best reflects the properties of a generic ideal M when M is *artinian*, i.e. M contains a power of each variable. Suppose this is the case for $M = \langle m_1, \ldots, m_r \rangle$, with $m_i = x_i^{d_i}$ for $i = 1, \ldots, n$.

Theorem 4.13 *The Scarf complex Δ_M of a generic artinian monomial ideal is a triangulation of the $(n-1)$-simplex with vertices $1, 2, \ldots, n$.*

It is not true that every triangulation occurs as the Scarf complex of a generic artinian monomial ideal. A first condition is that the triangulation be *regular*, which means that it is a certain kind of subcomplex of the boundary of an n-polytope. But even being regular is not enough; see [26]. In any case, regularity of Δ_M implies that the number of d-faces in the Scarf complex Δ_M is bounded above by the largest number of d-faces in any n-dimensional polytope with r vertices.

Corollary 4.14 *The number $\beta_i(M)$ of minimal i-th syzygies of M is bounded above by the maximum number $C_{i,n,r}$ of i-faces of any n-polytope with r vertices.*

In fact, the *Upper Bound Theorem* of P. McMullen (see [39, Theorem 8.23]) asserts that there is a polytope $C_n(r)$, the *cyclic polytope*, which simultaneously attains the maximum possible number $C_{i,n,r}$ of i-faces for each i. For $n < r$, the polytope $C_n(r)$ can be defined as the convex hull of any r distinct points on the curve $t \mapsto (t, t^2, \ldots, t^n)$. It turns out that the combinatorial type of $C_n(r)$ is independent of the choice of r points, and that the r points are precisely the vertices of the convex hull.

The bounds provided by cyclic polytopes are not sharp, but they are close: G. Agnarsson [1] has shown that $\beta_{i,n,r} \sim C_{i,n,r}$ asymptotically, for fixed n and i, as $r \to \infty$.

One can also ask for slightly less extreme behavior: call M *neighborly* if $\beta_1(M) = \binom{r}{2}$—that is, if every pair of generators is connected by a first syzygy. Hoşten and Morris [20] have shown that the maximum number of generators of a neighborly monomial ideal is

n	3	4	5	6	7	8	$\cdots$
r	4	12	81	2,646	1,422,564	229,809,982,112	$\cdots$

and they give a combinatorial expression for this number in general.

Example 4.15 ($n = 4$, $r = 12$) The generic monomial ideal

$$\begin{aligned} M &= \langle a^9, b^9, c^9, d^9, a^6b^7c^4d, a^2b^3c^8d^5, a^5b^8c^3d^2, \\ &\quad ab^4c^7d^6, a^8b^5c^2d^3, a^4bc^6d^7, a^7b^6cd^4, a^3b^2c^5d^8\rangle \end{aligned}$$

is neighborly. Its minimal free resolution is

$$0 \longleftarrow M \longleftarrow S^{12} \longleftarrow S^{66} \longleftarrow S^{108} \longleftarrow S^{53} \longleftarrow 0\,.$$

The irreducible decomposition of a generic artinian monomial ideal can be read off of the triangulation: a facet with label $\mathbf{x}^{\mathbf{a}}$ corresponds to an irreducible component $\mathfrak{m}^{\mathbf{a}}$. Note that $\mathbf{a}$ must have full support, simply because it is a facet of the Scarf complex of an artinian ideal. Using this idea, every generic monomial can have its irreducible decomposition read off of *some* Scarf complex. But which? Given any generic monomial ideal M, make it artinian by adding in high powers of the variables:

$$M^* := M + \langle x_1^D, x_2^D, \ldots, x_n^D\rangle, \qquad D \gg 0\,.$$

The reader should check that this operation preserves genericity.

Theorem 4.16 *Any generic monomial ideal M is the irredundant intersection of the ideals*

$$M_I := \langle x_i^{a_i} \mid a_i = \deg_{x_i}(m_I) < D\rangle\,,$$

where I runs over all facets of the triangulation Δ_{M^*}.

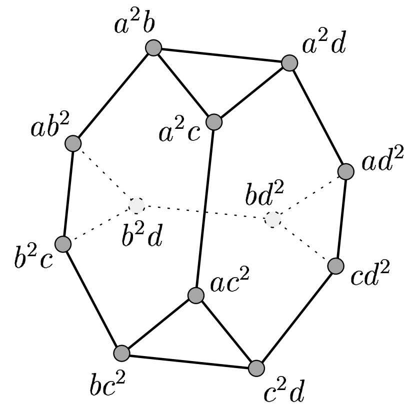
Example 4.17 Let $M = \langle x^3y^2z, x^2yz^3, xy^3z^2\rangle$ be the ideal from Section 3.3, but without any of the artinian generators. The irreducible decomposition of M is

$$M = \langle z\rangle \cap \langle y\rangle \cap \langle x\rangle \cap \langle y^2, z^3\rangle \cap \langle x^3, z^2\rangle \cap \langle x^2, y^3\rangle \cap \langle x^3, y^3, z^3\rangle\,.$$

The ideal in Section 3.3 plays the role of M^* here, and the reader should compare the irreducible decomposition here with the irreducible decomposition of M^* there. The white dots on the outside corners of the staircase diagram in Section 3.3 should be reexamined, as well. □

5 Lecture V: Cellular Resolutions

A number of times in the lectures up until now, we found that the constants in monomial matrices in free resolutions could be described in terms of geometric objects. Sometimes the geometric object seemed to come from nowhere (like Example 1.5), and sometimes from combinatorial data hidden in the generators and their least common multiples. Our aim in this lecture is to show how all monomial ideals "resolve themselves" via geometric resolutions, as suggested by the following picture. Here, the 12 vertices, 18 edges, and 8 faces in the polytope correspond to the Betti numbers 12, 18, and 8.



$$0 \longleftarrow S \longleftarrow S^{12} \longleftarrow S^{18} \longleftarrow S^{8} \longleftarrow S^{1} \longleftarrow 0$$

5.1 The basic construction

Let X be a *polyhedral cell complex* with r vertices (see [7, Section 6.2]). For instance, the cells of X may be the faces of a polytope, or a simplicial complex. Label each of the vertices of X by the generators $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_r}$ of a monomial ideal, and then label each face F of X by the least common multiple of the labels of its vertices:

$$\mathbf{x}^{\mathbf{a}_F} := \operatorname{lcm}(\mathbf{x}^{\mathbf{a}_i}, i \in F).$$

The exponent, $\mathbf{a}_F$ is called the *label* of the face F.

The polyhedral complex X is equipped with a *reduced chain complex*, which specializes to the usual reduced chain complex when X is simplicial.

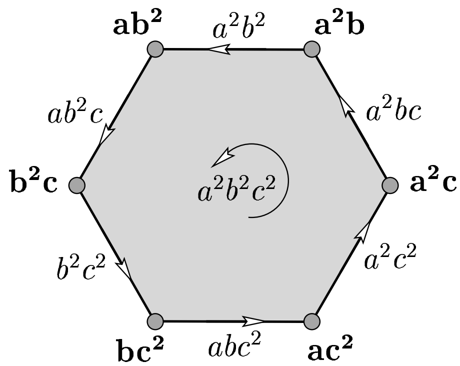
Definition 5.1 The *cellular free complex* $\mathbb{F}_X$ *supported on* X is the complex of $\mathbb{Z}^n$-graded free S-modules (with basis) given by monomial matrices as follows: use the boundary complex for X as the scalar entries, labeling the rows and columns by the face labels on X.

Without using monomial matrices, $\mathbb{F}_X$ and its boundary ∂ can be described as

$$\mathbb{F}_X = \bigoplus_{F \in X} S[-\mathbf{a}_F], \qquad \partial(F) = \sum_{\text{facets } G \text{ of } F} \operatorname{sign}(G, F)\, \frac{\mathbf{x}^{\mathbf{a}_F}}{\mathbf{x}^{\mathbf{a}_G}}\, G\,,$$

where F and G are thought of both as faces of X and as basis vectors in degrees $\mathbf{a}_F$ and $\mathbf{a}_G$. The sign for (G, F) is ± 1, and is part of the data in the boundary of the chain complex of X.

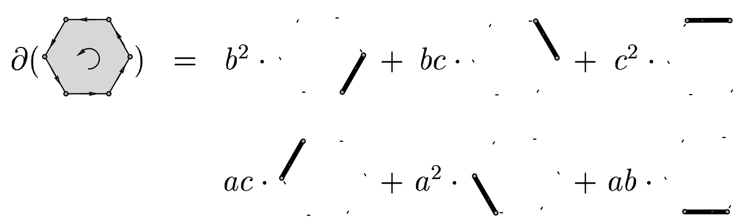
Example 5.2 The labeled hexagon



represents the complex $\mathbb{F}_X$, given by monomial matrices as:

$$
0 \leftarrow S \xleftarrow{\begin{array}{c|cccccc} & a^2c & a^2b & ab^2 & b^2c & bc^2 & ac^2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}} S^6 \xleftarrow{\begin{array}{c|cccccc} & a^2c^2 & a^2bc & a^2b^2 & ab^2c & b^2c^2 & abc^2 \\ a^2c & 1 & -1 & 0 & 0 & 0 & 0 \\ a^2b & 0 & 1 & -1 & 0 & 0 & 0 \\ ab^2 & 0 & 0 & 1 & -1 & 0 & 0 \\ b^2c & 0 & 0 & 0 & 1 & -1 & 0 \\ bc^2 & 0 & 0 & 0 & 0 & 1 & -1 \\ ac^2 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}} S^6 \xleftarrow{\begin{array}{c|c} & a^2b^2c^2 \\ a^2c^2 & 1 \\ a^2bc & 1 \\ a^2b^2 & 1 \\ ab^2c & 1 \\ b^2c^2 & 1 \\ abc^2 & 1 \end{array}} S \leftarrow 0
$$

The arrows denote the *orientations* of the faces, which determine the values of $\operatorname{sign}(G, F)$. For example, in the alternate way of writing cellular free complexes,



5.2 Exactness of cellular complexes

A subset $Q \subseteq \mathbb{Z}^n$ is an *order ideal* if $\mathbf{b} - \mathbf{a} \in Q$ whenever $\mathbf{b} \in Q$ and $\mathbf{a} \in \mathbb{N}^n$. Loosely, Q is "closed under going down" in the partial order on $\mathbb{Z}^n$. For an order ideal Q, we define a subcomplex of a labeled polyhedral complex X,

$$X_Q := \{F \in X \mid \mathbf{a}_F \in Q\}.$$

As a special case, for each $\mathbf{b} \in \mathbb{Z}^n$ define $X_{\preceq \mathbf{b}}$ to be the subcomplex of X consisting of faces whose degrees are coordinate-wise at most $\mathbf{b}$. For another example, say that $\mathbf{b}' \prec \mathbf{b}$ if $\mathbf{b}' \preceq \mathbf{b}$ and $\mathbf{b}' \neq \mathbf{b}$, and let $X_{\prec \mathbf{b}}$ be the subcomplex of X consisting of faces whose degrees are $\prec \mathbf{b}$.

Theorem 5.3 *$\mathbb{F}_X$ is exact if and only if $X_{\preceq \mathbf{b}}$ is acyclic over k (has no reduced homology) for all $\mathbf{b} \in \mathbb{N}^n$. In this case, $\mathbb{F}_X$ is called a* cellular resolution *of the monomial ideal M generated by its vertex labels, and the Betti numbers of M are*

$$\beta_{i,\mathbf{b}}(M) = \dim_k \widetilde{H}_{i-1}(X_{\prec \mathbf{b}}; k)\,.$$

Proof sketch: The free modules which contribute to the $\mathbb{Z}^n$-graded degree $\mathbf{b}$ part of the cellular free complex $\mathbb{F}_X$ are precisely those whose generators are in degrees $\preceq \mathbf{b}$; this proves the statement about acyclicity. The statement about Betti numbers comes from tensoring $\mathbb{F}_X$ with k. The resulting complex in degree $\mathbf{b}$ is the relative chain complex $\widetilde{\mathcal{C}}_{\bullet}(X_{\preceq \mathbf{b}}, X_{\prec \mathbf{b}}; k)$, whose homology is as claimed by the long exact sequence for relative homology, *because $X_{\preceq \mathbf{b}}$ is acyclic*—that is, because $\mathbb{F}_X$ is a resolution. □

Here is a nice example of how one can use the acyclicity criterion in a specific case. Recall that a polytope of dimension d is *simple* if every vertex meets d edges.

Proposition 5.4 *Let X be a simple polytope with facets $F_1, \ldots, F_n$ and vertices $v_1, \ldots, v_r$. Label each vertex v_i of X by the squarefree monomial $\prod_{v_i \notin F_j} x_j$. Then the labeled cell complex X supports a cellular resolution $\mathbb{F}_X$ of the monomial ideal I generated by the labels on its vertices. The resolution $\mathbb{F}_X$ is minimal and linear.*

Proof: A face G of X is contained in a facet F_j if and only if all of its vertices are contained in F_j. Hence, the label on G is

$$m_G = \frac{x_1 \cdots x_n}{\prod_{F_j \supseteq G} x_j} = \prod_{F_j \not\supseteq G} x_j\,.$$

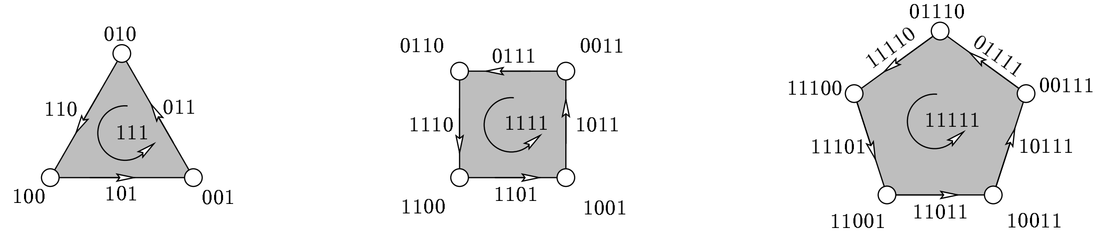
Acyclicity is easy to check: $X_{\preceq \mathbf{b}}$ is either empty, or it is the face of X obtained by intersecting all facets F_j for which x_j does not divide $x^{\mathbf{b}}$. In particular, $X_{\preceq \mathbf{b}}$ is contractible for all $\mathbf{b}$.

Minimality and linearity of $\mathbb{F}_X$ are both consequences of the fact that X is a simple polytope. Indeed, every face of a simple polytope is expressed uniquely as an intersection of facets. It follows from the equation for m_G that the labels on distinct faces G are unequal, which implies minimality. The formula for m_G also shows that m_G is a monomial of degree

$$\deg(m_G) = n - \dim(X) + \dim(G)$$

whence $\mathbb{F}_X$ is linear by the definition of the boundary map in $\mathbb{F}_X$. (Minimality also follows from linearity.) □

Example 5.5 When Q is two-dimensional, these resolutions follow the pattern



Note how the number of variables increases with the number of facets of Q. □

5.3 Examples of cellular resolutions

Taylor's resolution [36]. Let X be a simplex whose vertices are labeled by a set of monomials $\langle m_1, \dots, m_r \rangle$. The resulting cellular free complex is a cellular resolution of $M = \langle m_1, \dots, m_r \rangle$ called the *Taylor resolution*. Of course, the Taylor resolution tends not to be minimal: its length is r, and it has rank 2^r, and these should be compared with the bounds obtained using Scarf complexes in Lecture IV.

Scarf complex. If M is generic, then the minimal resolution of S/M is cellular, supported on the Scarf complex Δ_M.

CoScarf complex. There are ideals which are opposite to generic ideals, in a sense to be made precise in Lecture VI. They are called *cogeneric* monomial ideals, and their minimal resolutions are also cellular, supported on polyhedral complexes called *coScarf complexes.*

Permutohedron ideals. Let $\mathbf{u} = (u_1, u_2, \dots, u_n) \in \mathbb{Z}^n$ with $u_1 < u_2 < \dots < u_n$. By permuting the coordinates of $\mathbf{u}$, we obtain $n!$ points in $\mathbb{Z}^n \subset \mathbb{R}^n$ which are the vertices of an $(n-1)$-dimensional polytope called a *permutohedron* $P(\mathbf{u})$. The *permutohedron ideal* is the ideal $M(\mathbf{u})$ minimally generated by the monomials obtained by permuting the exponents of the monomial $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$. Labeling the vertices of the permutohedron with the generators of the permutohedron ideal in the natural way, we get a cellular resolution which is the minimal resolution of $M(\mathbf{u})$.

We now describe the degrees associated to each face of $P(\mathbf{u})$. Let $[n] := \{1, \dots, n\}$ and $\mathbf{v} \in \mathbb{R}^n$. For each $\sigma \subseteq [n]$, define $v_\sigma := \sum_{i \in \sigma} v_i$ and $\alpha_\sigma := \sum_{i=1}^{|\sigma|} u_i$. The permutohedron has the inequality description

$$P(\mathbf{u}) := \{\mathbf{v} \in \mathbb{R}^n \mid v_{[n]} = \alpha_{[n]} \text{ and } v_\sigma \geq \alpha_\sigma \text{ for all } \sigma \subset [n]\}.$$

Each i-dimensional face is determined by a chain of distinct proper subsets of $[n]$

$$\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_{n-i-1}$$

by setting $v_{\sigma_i} = \alpha_{\sigma_i}$ in the inequality description for $P(\mathbf{u})$. Given any such chain, define $\sigma_0 = \varnothing$ and $\sigma_{n-i} = [n]$. The label for the corresponding face F is then

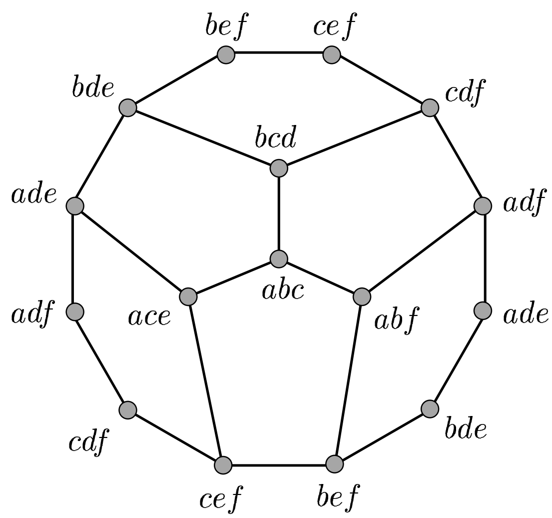
$$\mathbf{x}^{\mathbf{a}_F} = \prod_{j=1}^{n-i} \prod_{\ell \in \sigma_j \setminus \sigma_{j-1}} x_\ell^{\max\{\sigma_j \setminus \sigma_{j-1}\}}.$$

The hexagon in Example 5.2 is the minimal resolution of a permutohedron ideal $M(0, 1, 2)$. For more on the structure of permutohedra see [6], or check [23, Section 5] for more details on the connections with cellular resolutions.

Tree ideals. Let $M = \langle (\prod_{s\in I} x_s)^{n-|I|+1} \mid \varnothing \neq I \subseteq \{1,\ldots,n\}\rangle$ be the *tree ideal* in n variables (the name comes from the fact that M has the same number of standard monomials as there are labeled trees on $n+1$ vertices). Its minimal resolution is cellular, supported on the barycentric subdivision of an $(n-1)$-simplex. These will be investigated in Lecture VI, where we will find a relation to permutohedron ideals.

Irrelevant ideal of a toric variety. For a smooth (or just simplicial) projective toric variety there is a simple polytope (the moment polytope) whose vertices are labeled by the minimal generators of the so-called *irrelevant ideal of the Cox homogeneous coordinate ring* [9] as in Proposition 5.4. The corresponding cellular free complex is acyclic and minimal. We will say more about these ideals in Lecture VI.

The minimal triangulation of $\mathbb{RP}^2$. As in [7, Chapter 5], consider the Stanley-Reisner ideal of the minimal triangulation of the real projective plane. The cellular dual to the triangulation is a cell complex X consisting of six pentagons. We label the ten vertices of X with the minimal generators of the ideal:



If char $k \neq 2$, then X is acyclic, and the cellular complex $\mathbb{F}_X$ is the minimal free resolution

$$0 \longleftarrow S \longleftarrow S^{10} \longleftarrow S^{15} \longleftarrow S^{6} \longleftarrow 0\,.$$

If char $k = 2$, then X is not acyclic, and $\mathbb{F}_X$ is not a resolution of the ideal.

5.4 The hull resolution

Finally, our road leads to geometry: this section outlines a construction of Bayer and Sturmfels [4] which demonstrates how a canonical free resolution of a monomial ideal can be extracted from the way in which the exponents of the monomials sit inside of $\mathbb{R}^n$. Given $t \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{Z}^n$ set $t^{\mathbf{a}} := (t^{a_1},\ldots,t^{a_n}) \in \mathbb{R}^n$. For a monomial ideal M and $t \in \mathbb{R}$ consider the polyhedron $P_t := \mathrm{conv}\{t^{\mathbf{a}} \mid \mathbf{x}^{\mathbf{a}} \in M\}$.

Lemma 5.6 *The face poset of P_t is independent of t for $t \gg 0$. The vertices of P_t correspond precisely to the minimal generators of M.*

The polyhedron P_t is never bounded, since there are monomials of arbitrarily high degree in any (nonzero) monomial ideal.

Example 5.7 If $M = S$, so $S/M = 0$, then P_t is the shift by $(1, \dots, 1)$ of the real nonnegative orthant $\mathbb{R}^n_+$. If $M = \mathfrak{m} = \langle x_1, \dots, x_n \rangle$, then P_t is obtained from this shifted positive orthant by chopping off the vertex. □

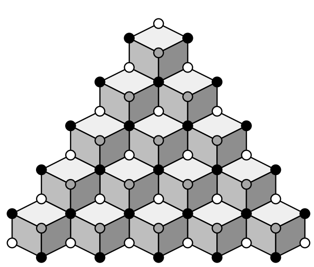
Definition 5.8 The *hull complex*, hull(M), is the polyhedral complex of bounded faces of P_t for $t \gg 0$. This complex is naturally labeled, with each vertex corresponding to a minimal generator of M.

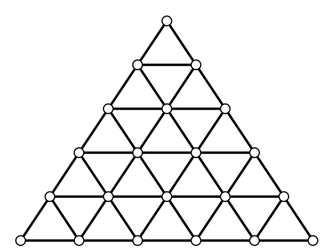
Theorem 5.9 *The cellular free complex $\mathbb{F}_{\text{hull}(M)}$ is a resolution of M.*

This *hull resolution* gives a canonical resolution of length at most n (the number of variables) for any monomial ideal, though it is usually non-minimal. Also, the free summands occurring in the hull resolution all have shifts which are least common multiples of generators, so that, for instance, the Betti numbers of a squarefree ideal are all in squarefree degrees.

Example 5.10

1. When $M = \mathfrak{m}$, the bounded faces are the simplex left by the truncated vertex in Example 5.7. This again yields a cellular description of the Koszul complex, as in Section 1.3.
2. The staircase diagram of $M = \langle x, y, z \rangle^5$ is at left below. For the hull resolution, look at the convex hull of $\{(t^i, t^j, t^k) \mid i + j + k = 5\}$ from the point $(1, \dots, 1)$, for $t > 1$.





The hull resolution of $\langle x, y, z \rangle^5$, at right above, respects the S_3-symmetry. In general, the hull resolution of $\langle x, y, z \rangle^d$ has two classes of second syzygies: the "up" triangles and the "down" triangles. The three edges of any "down" triangle have the same label (the coordinates of the black dots in the staircase), and are the reason for non-minimality: there should be two edges in each such degree.

It is always possible to remove edges from a cellular resolution of an ideal in three variables to get a minimal cellular resolution [27]. In this case, we have to remove one edge from each "down" triangle. When the power d is congruent to 0 or 1 mod 3, it is even possible to retain the S_3-symmetry [25].

3. The Scarf resolution of Section 5.3 is the hull resolution if M is generic! This is nontrivial (see [4]), and will not be proved here.
4. The minimal resolution of a permutohedron ideal (Section 5.3) is the hull resolution.
5. Not every minimal cellular resolution is a hull resolution. We will see systematic failures in the next lecture, provided by cogeneric monomial ideals. For a specific counterexample, the cellular resolution of the real projective plane in Section 5.3 is not the hull resolution. It is a good exercise to check this directly by writing down the hull complex explicitly, but there is a much easier reason: the hull complex is independent of characteristic, whereas the minimal resolution of the real projective plane is not.
6. In fact, the cellular resolutions of Proposition 5.4 are usually not hull resolutions, either. This is essentially because not every simple polytope has the same combinatorial type as the convex hull of a collection of squarefree vectors (i.e. vectors in $\{0,1\}^n$). □

6 Lecture VI: Alexander duality

The essence of Alexander duality for monomial ideals is the familiar optical illusion in which isometric drawings of cubes look alternately like they're pointing "in" or "out" (see Example 6.12). This duality generalizes the combinatorial notion of Alexander duality for simplicial complexes, where it is manifested quite simply as a switch between generators and irreducible components. More generally, this switch works on resolutions of monomial ideals: the data contained in the common multiples of generators is similar (but dual) to the data contained in the greatest common divisors of the irreducible components. The goal of this lecture is to define Alexander duality for ideals, and give a glimpse of how it is manifested in resolutions.

6.1 Squarefree monomial ideals

The *Alexander dual* $I^\vee$ of a squarefree monomial ideal I is obtained by switching the roles of the minimal generators and the prime components. A minimal generator of the form $\mathbf{x}^\sigma$ becomes the prime component $\mathfrak{m}^\sigma$. If $I = I_\Delta$ is the face ideal of Δ, then the simplicial complex $\Delta^\vee$ is defined by $I_{\Delta^\vee} = I_\Delta^\vee$. More directly, $\Delta^\vee = \{\overline{\tau} \mid \tau \notin \Delta\}$ consists of the complements of the nonfaces of Δ. It is straightforward (but very confusing for the beginner) to check that $(I^\vee)^\vee = I$. Perhaps the easiest way to make this check is by viewing Alexander duality as duality in the boolean lattice of subsets of $\{1,\ldots,n\}$.

Example 6.1 The ideals I_Δ and I_Γ of Examples 1.3 and 1.4 are Alexander dual; their generators and irreducible components are arranged to make this clear. □

Example 6.2 There are self-dual simplicial complexes, such as the two-dimensional simplicial complex consisting of an empty triangle and a single fourth vertex. There are also complexes which are isomorphic to their duals (after relabeling the vertices), but not equal. For example, the *stick twisted cubic* with ideal $I = \langle ab, bc, cd \rangle = \langle a, c \rangle \cap \langle b, c \rangle \cap \langle b, d \rangle$ has this property. $\square$

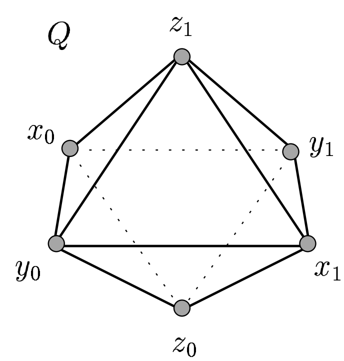
Example 6.3 Let Δ be the boundary of a simplicial d-polytope Q with n vertices $F_1, \ldots, F_n$ and r facets $v_1, \ldots, v_r$ (note the funny names). The irreducible decomposition of the face ideal I_Δ is

$$I_\Delta = \bigcap_{\text{facets } v_i} \langle x_j \mid F_j \notin v_i \rangle .$$

Since the facets of Q correspond to the vertices of the *polar polytope* X, and the vertices of Q correspond to the facets of X, the Alexander dual ideal $I_\Delta^\vee$ is the *irrelevant ideal* introduced in Section 5.3 and studied in Proposition 5.4.

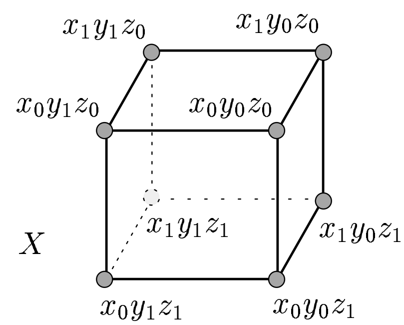
For example, let Q be the octahedron (the toric variety in question is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$)

$$\begin{aligned} I_\Delta &= \langle x_0, y_0, z_0 \rangle \cap \langle x_0, y_0, z_1 \rangle \cap \langle x_0, y_1, z_0 \rangle \cap \langle x_0, y_1, z_1 \rangle \\ &\quad \cap \langle x_1, y_0, z_0 \rangle \cap \langle x_1, y_0, z_1 \rangle \cap \langle x_1, y_1, z_0 \rangle \cap \langle x_1, y_1, z_1 \rangle \\ &= \langle x_0 x_1, y_0 y_1, z_0 z_1 \rangle \end{aligned}$$



whose vertices are labeled by variables x_i, y_i, or z_i depending on which axis they lie. The Alexander dual ideal is

$$\begin{aligned} I_\Delta^\vee &= \langle x_0 y_0 z_0, x_0 y_0 z_1, x_0 y_1 z_0, x_0 y_1 z_1, \\ &\qquad x_1 y_0 z_0, x_1 y_0 z_1, x_1 y_1 z_0, x_1 y_1 z_1 \rangle \\ &= \langle x_0, x_1 \rangle \cap \langle y_0, y_1 \rangle \cap \langle z_0, z_1 \rangle \end{aligned}$$



with the labeling described in Proposition 5.4 on the cube X polar to Q. The last intersection above explains why $I_\Delta^\vee$ is the irrelevant ideal of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. $\square$

Most readers will have encountered Alexander duality in a topological context, as an isomorphism between the homology of a closed topological subspace of a sphere and the cohomology of the complement. There is a straightforward connection with what we're doing here. A simplicial complex Δ is a closed subcomplex of the $(n-2)$-sphere which is the boundary of the simplex spanned by $\{1,\ldots,n\}$ (as long as $\{1,\ldots,n\}$ isn't a face of Δ). The complement of Δ in this sphere has a deformation retract to the Alexander dual simplicial complex $\Delta^\vee$. Therefore, the topological Alexander duality relation holds between the homology and cohomology of these simplicial complexes:

Theorem 6.4 $\widetilde{H}_{i-1}(\Delta^\vee;k) \cong \widetilde{H}^{n-2-i}(\Delta;k)$.

Proof: Here's a fun proof which is also quite apropos. As in the proof of Theorem 1.8, let's calculate $\beta_{i,\mathbf{1}}(I_\Delta)$ using the symmetry of Tor, where $\mathbf{1} = (1,\ldots,1)$. Only this time, we tensor the Koszul complex $\mathbb{K}_{\bullet}$ (whose scalars are the *chain* complex of a simplex rather than the cochain complex) with I_Δ (rather than with S/I_Δ). The $\mathbb{Z}^n$-graded degree $\mathbf{1}$ part $(I_\Delta \otimes \mathbb{K}_{\bullet})_{\mathbf{1}}$ is a chain subcomplex of $(\mathbb{K}_{\bullet})_{\mathbf{1}} = \widetilde{\mathcal{C}}_{\bullet}(\{1,\ldots,n\};k)$ generated by some of the basis vectors e_σ, and is therefore the reduced chain complex for some simplicial complex Γ. The isomorphism $(\mathbb{K}_{\bullet})_{\mathbf{1}} \to \widetilde{\mathcal{C}}_{\bullet}((n-1)\text{-simplex};k)$ takes $\mathbf{x}^{\overline{\sigma}} \cdot 1_\sigma \mapsto e_\sigma$, so

$$\sigma \in \Gamma \iff \mathbf{x}^{\overline{\sigma}} \otimes 1_\sigma \in (I_\Delta \otimes \mathbb{K}_{\bullet})_{\mathbf{1}} \iff \mathbf{x}^{\overline{\sigma}} \in I_\Delta \iff \sigma \in \Delta^\vee$$

whence $\Gamma = \Delta^\vee$. Since $\varnothing$ sits in homological degree 0 instead of -1, the i^{th} homology of $(I_\Delta \otimes \mathbb{K}_{\bullet})_{\mathbf{1}}$ is $\widetilde{H}_{i-1}(\Delta^\vee;k)$. Comparison with Theorem 1.8 completes the proof. □

Remark 6.5 This proof doesn't use any properties of k, and can be used over $\mathbb{Z}$ or any other ring, since the Koszul complex still resolves $\mathbb{Z}$ as a $\mathbb{Z}[x_1,\ldots,x_n]$-module. □

We only needed the degree $\mathbf{1}$ part of $I_\Delta \otimes \mathbb{K}_{\bullet}$ in the theorem above, but more generally, we can try calculating the squarefree degree σ parts. We find that $(I_\Delta \otimes \mathbb{K}_{\bullet})_\sigma \cong \widetilde{\mathcal{C}}_{\bullet}(\mathrm{link}_{\overline{\sigma}}\Delta^\vee;k)$ is the reduced chain complex for the *link of* $\overline{\sigma}$ *in* $\Delta^\vee$:

$$\mathrm{link}_{\overline{\sigma}}\Delta^\vee := \{\tau \in \Delta^\vee \mid \tau \cup \overline{\sigma} \in \Delta^\vee \text{ and } \tau \cap \overline{\sigma} = \varnothing\}.$$

Therefore, the Alexander duality theorem for simplicial complexes is really an application of the fact that Hochster's theorem has two possible statements:

Theorem 1.8 *(Dual version)* $\beta_{i,\sigma}(I_\Delta) = \beta_{i+1,\sigma}(S/I_\Delta) = \dim_k \widetilde{H}_{i-1}(\mathrm{link}_{\overline{\sigma}}\Delta^\vee;k)$.

The equality between (co)homology groups in the dual statements of Theorem 1.8 suggests the following enhancement of Theorem 6.4:

Corollary 6.6 *The simplicial complexes* $\mathrm{link}_\sigma\Delta$ *and* $\Delta^\vee|_{\overline{\sigma}}$ *are Alexander dual* inside the simplex $\overline{\sigma}$. *Therefore,*

$$\widetilde{H}^{|\sigma|-2-i}(\Delta|_\sigma;k) \cong \widetilde{H}_{i-1}(\mathrm{link}_{\overline{\sigma}}\Delta^\vee;k).$$

In this formula, we can switch (i) $|\sigma|-2-i$ *and* $i-1$*; (ii)* $\widetilde{H}_{\bullet}$ *and* $\widetilde{H}^{\bullet}$*; (iii)* Δ *and* $\Delta^\vee$*; or (iv)* σ *and* $\overline{\sigma}$*.*

Note that if $\sigma \notin \Delta$ then $\mathrm{link}_\sigma\Delta = \{\}$ = the void complex while $\Delta^\vee|_{\overline{\sigma}}$ is the entire simplex $\overline{\sigma}$ with all its faces, and all of the (co)homology in the Corollary vanishes.

Theorems of Eagon-Reiner and Terai. The interaction of Alexander duality with commutative algebra has received a lot of attention recently, sparked in large part by a theorem of Eagon-Reiner [11], and its generalization by Terai [37]. Recall that the length of the minimal resolution of S/I_Δ is the *projective dimension* of S/I_Δ, denoted $\mathrm{pd}(S/I_\Delta)$. This number is at least the codimension of I_Δ, which equals the smallest number of generators of any irreducible component. Equality holds if and only if S/I_Δ is *Cohen-Macaulay.* By the Hilbert syzygy theorem, the projective dimension is no larger than n.

On the other hand, the *regularity* of I_Δ is a measure of how "wide" the free resolution is:

$$\mathrm{reg}(I_\Delta) := \max\{|\sigma| - i \mid \beta_{i,\sigma}(I_\Delta) \neq 0\}.$$

The regularity is at least the degree of the smallest generator of I_Δ, and I_Δ is said to have *linear free resolution* if equality holds. The duality theorem of Eagon and Reiner says that the conditions of minimality in the regularity and projective dimension are Alexander dual. Terai generalized this to say, in addition, that the defect from achieving minimality is also transferred by Alexander duality ("length is dual to width").

Theorem 6.7 *S/I_Δ is Cohen-Macaulay if and only if $I_\Delta^\vee$ has linear free resolution [11]. Moreover,* $\mathrm{pd}(S/I_\Delta) = \mathrm{reg}(I_\Delta^\vee)$ *[37].*

Their proofs worked by careful use of Hochster's theorem 1.8 as well as its dual version, and we will not reproduce them here. We do note, however, that the first statement follows immediately from the second: this is because the codimension of I_Δ is obviously equal to the smallest degree of a generator of $I_\Delta^\vee$, by the very definition of Alexander dual ideal ($\mathfrak{m}^\sigma \leftrightarrow \mathbf{x}^\sigma$).

There are lots of useful criteria for determining when a face ideal is Cohen-Macaulay. We will mention two in the examples below, but we say nothing more, partly because there is a vast literature on the subject. See, for instance, [32] or [7, Chapter 5].

Example 6.8 The face ideal of any simplicial sphere Δ is Cohen-Macaulay. In particular, if Δ is the boundary of a simplicial polytope Q as in Example 6.3, then I_Δ is Cohen-Macaulay. By Theorem 6.7, $I_\Delta^\vee$ has a linear resolution. Of course, we already know from Proposition 5.4 that this linear resolution is cellular, supported on the polar polytope X. See Example 6.3 for a picture of this linear resolution. □

Example 6.9 The stick twisted cubic of Example 6.2 is Cohen-Macaulay because the simplicial complex is 1-dimensional and connected. On the other hand, we found that the Alexander dual of the stick twisted cubic is just another stick twisted cubic, and therefore also Cohen-Macaulay. Thus Theorem 6.7 implies that the face ideal has a linear resolution, as well.

6.2 Arbitrary monomial ideals

For squarefree monomial ideals, Alexander duality can be confusing, with too many $\{0,1\}$ vectors and subsets of $\{1,\ldots,n\}$ flying around along with their complements. When the duality is generalized to arbitrary monomial ideals, however, the confusion subsides a little, as the various squarefree vectors begin to take different roles: we are forced to forgo our conventions of automatically identifying any two objects representing a subset of $\{1,\ldots,n\}$.

Of course, the definition of Alexander dual must necessarily become more complicated. Nonetheless, the basic idea remains the same: make the irreducible components into generators. Each monomial ideal has two unique irredundant representations, as a sum principal ideals or as an intersection of irreducible ideals:

$$\begin{aligned} I &= \langle \mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}}, \ldots, \mathbf{x}^{\mathbf{c}} \rangle \\ &= \mathfrak{m}^{\mathbf{u}} \cap \mathfrak{m}^{\mathbf{v}} \cap \ldots \cap \mathfrak{m}^{\mathbf{w}} \end{aligned}$$

(recall the notation from Section 0.6). This is analogous to the situation with convex polytopes, which can be described either as a convex hull of vertices or as an intersection of bounding halfplanes.

Let $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$. As usual, we write $\mathbf{a} \succeq \mathbf{b}$ if $a_i \geq b_i$ for $i = 1, \ldots, n$, and in that case we define $\mathbf{a} \smallsetminus \mathbf{b}$ to be the vector whose i-th coordinate is $a_i + 1 - b_i$ if $b_i \geq 1$ and 0 otherwise. For example, $(7,6,5) \smallsetminus (2,0,3) = (6,0,3)$. Now we can make the following definition, due to E. Miller [23].

Definition 6.10 Let I be a monomial ideal and $\mathbf{a}_I \in \mathbb{N}^n$ the exponent vector on the least common multiple of the minimal generators of I. For $\mathbf{a} \succeq \mathbf{a}_I$, define the *Alexander dual of I with respect to* $\mathbf{a}$ by

$$I^{[\mathbf{a}]} := \langle \mathbf{x}^{\mathbf{a} \smallsetminus \mathbf{b}} \mid \mathfrak{m}^{\mathbf{b}} \text{ is an irreducible component of } I \rangle \,.$$

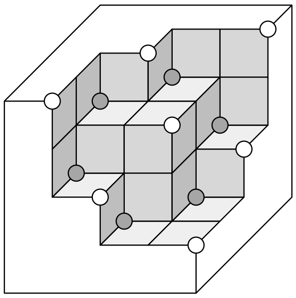
If $\mathbf{a} = \mathbf{a}_I$, let $I^{\vee} := I^{[\mathbf{a}_I]}$ be called simply the *Alexander dual* of I.

Example 6.11 Let $\mathbf{a} = (4,4,4)$. Then

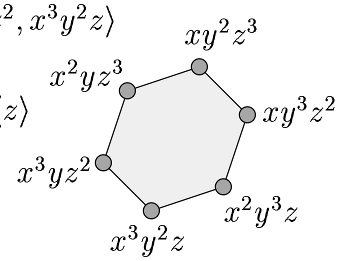
$$\begin{aligned} I &= \langle x^3, xy, yz^2 \rangle \\ &= \langle x^3, y \rangle \cap \langle x, z^2 \rangle \end{aligned} \quad \Longrightarrow \quad \begin{aligned} I^{[\mathbf{a}]} &= \langle x^2 \rangle \cap \langle x^4, y^4 \rangle \cap \langle y^4, z^3 \rangle \\ &= \langle x^2y^4, x^4z^3 \rangle \,. \end{aligned}$$

Note that $(I^{[\mathbf{a}]})^{[\mathbf{a}]} = I$; see Corollary 6.15. □

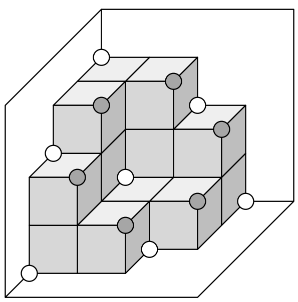
Example 6.12 *(Permutohedron ideal and tree ideals.)*
The optical illusion of Alexander duality is evident in 3-dimensional staircase diagrams. The permutohedron ideal



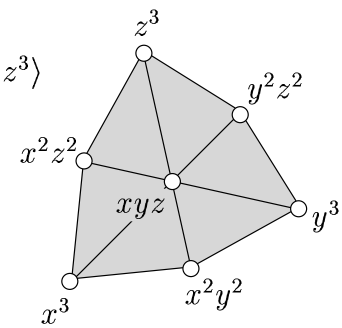
$$I = \langle xy^2z^3, xy^3z^2, x^2yz^3, x^2y^3z, x^3yz^2, x^3y^2z\rangle$$
$$= \langle x^3, y^3, z^3\rangle \cap \langle x^2, y^2\rangle \cap \langle x^2, z^2\rangle \cap \langle y^2, z^2\rangle \cap \langle x\rangle \cap \langle y\rangle \cap \langle z\rangle$$



is Alexander dual to the tree ideal



$$I^\vee = \langle xyz, x^2y^2, x^2z^2, y^2z^2, x^3, y^3, z^3\rangle$$
$$= \langle x^3, y^2, z\rangle \cap \langle x^3, y, z^2\rangle \cap \langle x^2, y^3, z\rangle \cap \langle x^2, y, z^3\rangle \cap \langle x, y^3, z^2\rangle \cap \langle x, y^2, z^3\rangle$$



with respect to the vector $\mathbf{a} = (3, 3, 3)$. To convince yourself of the duality, turn the page upside-down, and compare the pair of staircase diagrams—there is no difference. In fact, a shortcut in producing these diagrams is to have the drawing program (`xfig`) turn the first staircase upside-down. Note how dots of the same color correspond in the two staircases.

We have already remarked in Section 5.3 that the minimal free resolutions of these ideals are cellular. For later reference, these resolutions are drawn at right. A more general family of examples will be introduced in Section 6.4, below. □

One of the nice things about squarefree Alexander duality is that it takes a set of *minimal* generators to a set of *irredundant* irreducible components. If our general version is to do the same thing, a prerequisite is that it take incomparable irreducible ideals to incomparable monomials.

Lemma 6.13 $\mathbf{x}^{\mathbf{a}\smallsetminus\mathbf{b}}$ *divides* $\mathbf{x}^{\mathbf{a}\smallsetminus\mathbf{c}}$ *if and only if* $\mathfrak{m}^{\mathbf{b}} \subseteq \mathfrak{m}^{\mathbf{c}}$.

Using this one can show that the generating set for $I^{[\mathbf{a}]}$ in Definition 6.10 is minimal.

The notation $I^\vee$ introduced above is consistent with its earlier use in the squarefree case: $I_\Delta^\vee = I_{\Delta^\vee}$ because $\mathbf{b}^{(1,\ldots,1)} = \mathbf{b}$ for $\mathbf{b} \in \{0,1\}^n$. In that case we found that we could switch either the generators for irreducible components, or vice-versa. The same holds here, as shown by Miller [23]:

Theorem 6.14 *With notation as in the definition,*

$$I^{[\mathbf{a}]} = \bigcap \{\mathfrak{m}^{\mathbf{a}\smallsetminus\mathbf{b}} \mid \mathbf{x}^{\mathbf{b}} \textit{ is a minimal generator of } I\}.$$

In the squarefree case, we also found that $(I^\vee)^\vee = I$. That is no longer true here (!), see Exercise A.3.1. But if we fix $\mathbf{a}$, we have an honest duality.

Corollary 6.15 $(I^{[\mathbf{a}]})^{[\mathbf{a}]} = I$ *for any* $\mathbf{a} \succeq \mathbf{a}_I$.

6.3 Duality for resolutions

There are many kinds of duality in commutative algebra and algebraic geometry, and it seems that they're all, in their own way, repercussions of the adjointness of Hom and $\otimes$, or more simply, the ordinary duality between a vector space and its dual. Alexander duality is no exception. It is possible to extend the definition of Alexander duality to arbitrary $\mathbb{N}^n$-graded modules, and even to resolutions, both free and *injective.* Instead of developing this theory here, we give in this section some numerical consequences, including the generalization to arbitrary monomial ideals by Miller of the theorems of Eagon-Reiner and Terai. An extended geometric example occupies the next section. Further theory can be found in [23] for monomial ideals, or [24] for resolutions, especially injective resolutions, which have an elegant and concrete description in the $\mathbb{Z}^n$-graded situation.

The next theorem can be thought of as the reflection for arbitrary monomial ideals of the fact that Hochster's theorem 1.8 has two equivalent and dual statements. In the case where $I = I_\Delta$ and $\mathbf{a} = (1, \ldots, 1)$, it reduces to the combinatorial Alexander duality theorem 6.4.

Theorem 6.16 *If* $\mathbf{b} \in \mathbb{N}^n_+$ *is a vector of strictly positive integers, then* $\beta_{i,\mathbf{b}}(I) = \beta_{n-i-1,\mathbf{a}\smallsetminus\mathbf{b}}(I^{[\mathbf{a}]})$.

Example 6.17 The table below lists some instances where the Betti numbers are 1 for the permutohedron and tree ideals of Example 6.12, where $\mathbf{a} = (3, 3, 3)$:

i	$\mathbf{b}$	$3-i-1$	$\mathbf{a}\smallsetminus\mathbf{b}$
0	$(1,2,3)$	2	$(3,2,1)$
1	$(1,3,3)$	1	$(3,1,1)$
2	$(3,3,3)$	0	$(1,1,1)$

$$\beta_{i,\mathbf{b}}(I) = \beta_{n-i-1,\mathbf{a}\smallsetminus\mathbf{b}}(I^{[\mathbf{a}]}) = 1, \quad \mathbf{a} = (3,3,3)$$

As you look at the pictures in Example 6.12 to verify these equalities, note both the positions of these degrees in the staircases, and which are the corresponding faces in the cellular resolutions. □

Theorem 6.7 can't hold verbatim for arbitrary monomial ideals, since one side of the equality (projective dimension) is bounded, while the other (regularity) is not.

On the other hand, regularity is not a particularly $\mathbb{Z}^n$-graded thing to measure—the usual definition requires us to sum the coordinates of the degree $\mathbf{b}$, which is more of a $\mathbb{Z}$-graded procedure. The generalization of the Eagon-Reiner and Terai theorems needs a $\mathbb{Z}^n$-graded analog of regularity.

Definition 6.18 The *support-regularity* of a monomial ideal I is

$$\operatorname{supp.reg}(I) \quad := \quad \max\{|\operatorname{supp}(\mathbf{b})| - i \mid \beta_{i,\mathbf{b}}(I) \neq 0\},$$

and I is said to have a *support-linear free resolution* if there is a $d \in \mathbb{N}$ such that $|\operatorname{supp}(m)| = d = \operatorname{supp.reg}(I)$ for all minimal generators m of I.

For squarefree ideals the notions of regularity and support-regularity coincide, since the only degrees we ever care about are squarefree. In particular, the following theorem of Miller [24] specializes to those of Eagon-Reiner and Terai when $\mathbf{a} = (1, \ldots, 1)$.

Theorem 6.19 *Let $\mathbf{a} \succeq \mathbf{a}_I$. Then S/I is Cohen-Macaulay if and only if $I^{[\mathbf{a}]}$ has a support-linear free resolution. More generally,* $\operatorname{pd}(S/I) = \operatorname{supp.reg}(I^{[\mathbf{a}]})$.

As in Theorem 6.7, the second statement implies the first, and for the same reason. The proof can be accomplished using equalities similar to those in Theorem 6.16 but with *Bass numbers*, which convey the numerics of injective resolutions just as Betti numbers do for free resolutions. Essentially, the decreases of the dimensions of the indecomposable injective summands in a minimal injective resolution of S/I correspond precisely to the increases in the supports of the degrees in the minimal free resolution of $I^{[\mathbf{a}]}$, and the former detect the projective dimension of S/I.

6.4 Cogeneric monomial ideals

A monomial ideal M is *cogeneric* if its Alexander dual $M^\vee$ is generic, which happens if and only if $M^{[\mathbf{a}]}$ is generic for all $\mathbf{a}$ as in Definition 6.10. This condition can be translated into a direct characterization in terms of irreducible components. Let $M = M_1 \cap \cdots \cap M_r$ be an irredundant irreducible decomposition.

Definition 6.20 A monomial ideal is *cogeneric* if, whenever distinct irreducible components M_i and M_j have a minimal generator in common, there is an irreducible component $M_\ell \subset M_i + M_j$ such that M_ℓ and $M_i + M_j$ do not have a minimal generator in common.

Example 6.21 The permutohedron ideal (Example 6.12 and Section 5.3) is cogeneric. It is perhaps a little easier to check that its Alexander dual, the tree ideal, is generic. Note, however, that the tree ideal is not strongly generic. □

In what follows, let $D \gg 0$ be a fixed large integer, and $\mathbf{D} = (D, \ldots, D)$. The generators of a cogeneric monomial ideal M correspond to the irreducible components of the generic monomial ideal $M^{[\mathbf{D}]}$. We have already seen in Lecture IV that such irreducible components can be detected from the Scarf complex, after we add in high powers of the variables. Therefore, let $M^* = M^{[\mathbf{D}]} + \langle x_1^{D+1}, \ldots, x_n^{D+1} \rangle$, so that the facets of the Scarf complex Δ_{M^*} correspond to the generators of M.

We are going to make a free resolution of M out of Δ_{M^*} which is related to the algebraic Scarf complex $\mathbb{F}_{\Delta_{M^*}}$ in much the same way that the "coKoszul complex" $\mathbb{K}^{\bullet}$ is related to the usual Koszul complex $\mathbb{K}_{\bullet}$ in Section 1.3. Consider the labels on the Scarf complex as exponent vectors rather than monomials, and subtract each label from $(D+1, \ldots, D+1)$. Now make a complex of free S-modules by using the *coboundary* complex of Δ_{M^*} for scalars in monomial matrices, with the new labels from Δ_{M^*} on the rows and columns. Then take the submatrices whose rows and columns are indexed by *interior* faces of Δ_{M^*}. More succinctly, the scalars are the *relative cochain complex* of the pair $(\Delta_{M^*}, \partial\Delta_{M^*})$, where "$\partial$" means "boundary of".

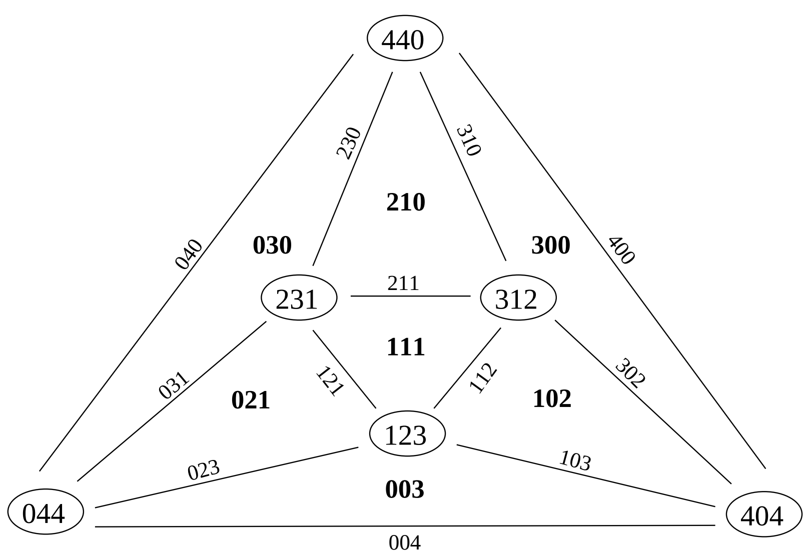
Theorem 6.22 *Suppose M is cogeneric. The monomial matrix whose scalars are the cochain complex of the interior int(Δ_{M^*}), with row and column labels as described above, is the minimal free resolution of M, called the* coScarf resolution.

This theorem was proved in various forms and strengths in [34, 23, 26]. The reader should consult [26] for more on cogeneric monomial ideals. As an application we now have an algorithm for intersecting a generic collection of irreducible ideals: compute Δ_{M^*}, and read off the facet labels!

Example 6.23 To compute the generators and resolution of

$$M = \langle x, y^2, z^3 \rangle \cap \langle x^2, y^3, z \rangle \cap \langle x^3, y, z^2 \rangle ,$$

first draw the Scarf complex Δ_{M^*} for $M^* = M^{[\mathbf{D}]} + \langle x^{D+1}, y^{D+1}, z^{D+1} \rangle$; this has been done in Section 3.3, with $D = 3$. Now relabel according to the regimen above, subtracting all of the face labels from $(4, 4, 4)$:



Reading the facet labels tells us that

$$M = \langle z^3, y^3, x^3, y^2z, xz^2, x^2y, xyz \rangle ,$$

and restricting the cochain complex to the interior faces yields the minimal free resolution

$$0 \longleftarrow S^7 \longleftarrow S^9 \longleftarrow S^3 \longleftarrow 0$$

corresponding to 3 vertices, 9 edges, and 7 facets. □

Remark 6.24 The coScarf resolution is rarely the hull resolution of a cogeneric monomial ideal. It can happen, though, that they coincide, as they do for the permutohedron ideal. □

The coScarf resolution, though it is defined here to be "relative cocellular", is actually cellular, as well (like $\mathbb{K}^\bullet$ and $\mathbb{K}_\bullet$), as is evident for the permutohedron. To get the general idea, superimpose the hexagon (cellular version of the coScarf resolution for cogeneric permutohedron ideal) onto the barycentric subdivision of the triangle (Scarf resolution of generic tree ideal).

Just as the Scarf resolution is a special case of the hull resolution, the coScarf resolution is a special case of the *cohull resolution* [23], which does the same thing to the hull resolution of an arbitrary artinian monomial ideal as the coScarf resolution does to the Scarf resolution when the ideal is generic. In the cohull construction, the hull complex of the Alexander dual ideal $M^\vee$ is defined by its facets, which are labeled by the generators of M, while the vertices correspond to the irreducible components of M. This contrasts with the usual case, where the facets of the hull complex of M represent the irreducible components of M, while the vertices represent the minimal generators. Thus we complete our tour of Alexander duality, by showing how the statement about minimal generators vs. irreducible components and vertices vs. bounding half-spaces is really a deep truth rather than a convenient analogy: it is the principle underlying the interaction of Alexander duality with the geometric constructions of cellular resolutions.

7 Lecture VII: Monomial modules to lattice ideals

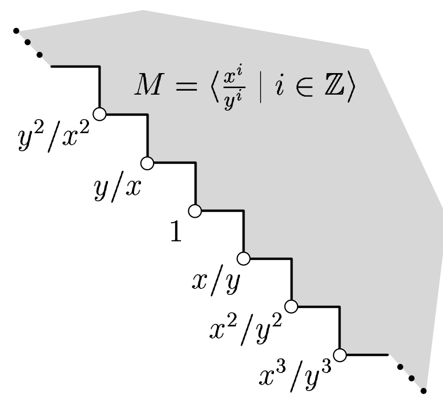
In this lecture and the next we study certain kinds of limits of monomial ideals. Here the limit will have an action of a *lattice* L—that is, a finitely generated subgroup of $\mathbb{Z}^n$. This establishes a relation between monomial ideals and binomial ideals: the quotient by L is a *lattice ideal*, of which a *toric ideal* is a special case. The cellular resolutions and genericity conditions of previous lectures will also prove to be of use. The general theory in this lecture was developed by D. Bayer and B. Sturmfels in [4], although genericity for lattice ideals is due to Sturmfels in collaboration with I. Peeva, and S. Popescu joined Bayer and Sturmfels for the work on Lawrence ideals [3].

7.1 Monomial modules

The Laurent polynomial ring $T = S[x_1^{-1}, \ldots, x_n^{-1}]$ is a module over S. An S-submodule M of T which is generated by Laurent monomials, $\mathbf{x}^{\mathbf{a}}$ with $\mathbf{a} \in \mathbb{Z}^n$, is called a *monomial module.* In general, such modules need not be minimally generated, but every monomial module M in this lecture will have a minimal generating set of monomials. If this set is finite, then M is a $\mathbb{Z}^n$-graded shift of a monomial ideal of S.

We can still draw pictures, but the usual staircase diagrams for monomial ideals become infinite staircases for monomial modules.

Example 7.1 Consider the monomial module in $k[x,y][x^{-1},y^{-1}]$ generated by the Laurent monomials $(\frac{x}{y})^i$ for $i \in \mathbb{Z}$. The staircase diagram really is a staircase, but an infinite one:



This monomial module can be thought of as the "mother of all powers of the maximal ideal": intersecting it with any shift of S produces a power of the maximal ideal. More generally, a monomial module can be thought of as the limit of the monomial ideals obtained by intersecting it with shifted positive orthants. □

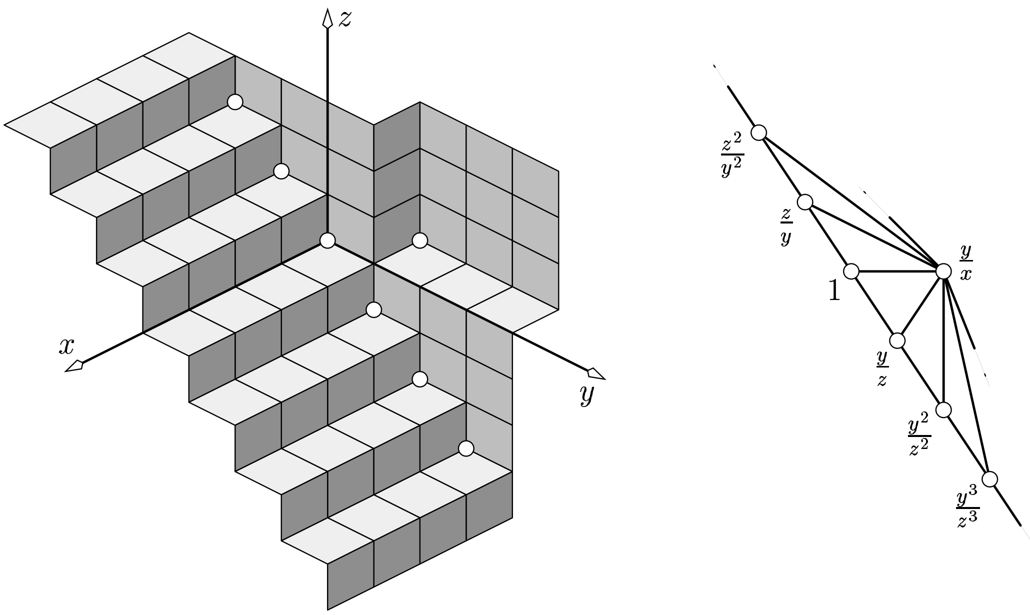
The construction of the hull complex in Section 5.4 works here mutatis mutandis: fix $t \gg 0$, and form the unbounded n-dimensional polyhedron $P_t = \operatorname{conv}\{t^{\mathbf{a}} \mid \mathbf{x}^{\mathbf{a}} \in M\}$. The combinatorial type is independent of the large real number t, and its vertices are precisely the minimal generators of M (this is why we are assuming M has minimal generators). Again, a complex of free modules $\mathbb{F}_{\text{hull}(M)}$ is defined, and the main result still holds; in fact, this is the generality in which it was originally published [4].

Theorem 7.2 *$\mathbb{F}_{\text{hull}(M)}$ is a free resolution of M.*

Example 7.3 The hull complex hull(M) for the monomial module in the previous example is the real line with a vertex at each integer point. □

Example 7.4 A cell complex is said to be *locally finite* if every face meets finitely many others. In general, the hull complex of a monomial module need not be locally finite. For example, consider the monomial module M in three variables x, y, z:

$$M = \langle \tfrac{y}{x} \rangle + \langle (\tfrac{y}{z})^i \mid i \in \mathbb{Z} \rangle\,.$$



The vertex $\frac{y}{x}$ lies on infinitely many triangles of hull(M). □

7.2 Lattice modules

In the all of the lectures before this one, we have been interested in monomial modules whose generating sets are finite. Here, we consider another interesting case, when the set of generating Laurent monomials forms a group under multiplication. To be precise, let $L \subset \mathbb{Z}^n$ be a sublattice whose intersection with $\mathbb{N}^n$ is $\{\mathbf{0}\}$. This ensures the existence of a functional with strictly positive coordinates that vanishes on L.

Definition 7.5 The *lattice module* M_L is the S-submodule of $T = S[x_1^{-1}, \ldots, x_n^{-1}]$ generated by $\{\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in L\}$.

The hypothesis on L guarantees that the elements of L form a *minimal* generating set for M_L.

Example 7.6 The monomial module in Example 7.1 is actually the lattice module M_L for the lattice $\ker(1,1)$. More generally, the "mother of all maximal ideals" in n variables is the lattice module for $L = \ker(1,\ldots,1)$. For a picture of a finite part of the staircase of $\ker(1,1,1)$, where $n = 3$, see Example 5.10. □

Because the generating monomials form a group L, that group acts on the monomial module M_L, the action being given by $\mathbf{a} \cdot \mathbf{x}^{\mathbf{b}} = \mathbf{x}^{\mathbf{a}+\mathbf{b}}$ for $\mathbf{a} \in L$ and $\mathbf{x}^{\mathbf{b}} \in M_L$. What would be really nice is if we could find a whole free resolution of M_L which is acted on by L. Such an *equivariant free resolution* is provided by the hull resolution. The point is that the lattice L permutes the faces of hull(M).

Example 7.7 The hull complex of the "mother of all maximal ideals" in two variables (Example 7.3) is obviously acted on by $L \cong \mathbb{Z}$. In three variables, the hull complex is the tesselation of $\mathbb{R}^2$ by triangles, a part of which is depicted in Example 5.10. Note that in the case of three variables, the action of $L = \ker(1,1,1)$ on the hull complex is easiest to see in the picture if we view the tesselation as actually sitting in $\mathbb{R} \otimes L \subset \mathbb{R}^3$ instead of in the plane. □

Calculating the hull complex is actually a finite thing to do (although it is an open problem to determine an efficient algorithm), even though it has infinitely many cells. This is because of the following minor miracle:

Proposition 7.8 *The hull complex of a lattice module is locally finite.*

In fact, one produces an explicit finite set of edges (a *Graver basis* for L) such that the vertex $\mathbf{0} \in L$ meets at most these edges, and hence $\mathbf{0}$ meets finitely many faces altogether. But every vertex of $\text{hull}(M_L)$ is equivalent to $\mathbf{0}$ under the action of L, so we only need the local behavior of $\text{hull}(M_L)$ around $\mathbf{0}$ to determine the entire hull complex. Thus we reduce the calculation of $\text{hull}(M_L)$ to the computation of $\text{hull}(I)$ for a monomial module I generated by a finite set of elements of L. In other words, the hull complex of a monomial ideal is enough information to determine $\text{hull}(M_L)$ *modulo the action of L on the faces*, which has only finitely many orbits.

7.3 Genericity

A monomial module $M \subset T$ is *generic* if all its minimal first syzygies $\mathbf{x}^{\mathbf{a}}\mathbf{e}_i - \mathbf{x}^{\mathbf{b}}\mathbf{e}_j$ have full support, i.e. every variable x_ℓ appears either in $\mathbf{x}^{\mathbf{a}}$ or $\mathbf{x}^{\mathbf{b}}$. This definition is the essence behind genericity for monomial ideals, although for ideals there are "boundary effects" coming from the fact that $\mathbb{N}^n$ is a special subset of $\mathbb{Z}^n$. To be precise, the genericity condition on the minimal first syzygies $\mathbf{x}^{\mathbf{a}}\mathbf{e}_i - \mathbf{x}^{\mathbf{b}}\mathbf{e}_j$ of an ideal requires only that $\text{supp}(\mathbf{x}^{\mathbf{a}+\mathbf{b}}) = \text{supp}(\text{lcm}\,(m_i, m_j))$, as opposed to $\text{supp}(\mathbf{x}^{\mathbf{a}+\mathbf{b}}) = \{1, \ldots, n\}$ for monomial modules. This definition allows us to treat the boundary exponent 0 differently than the strictly positive exponents coming from the interior of $\mathbb{N}^n$. Just like the hull complex, the Scarf complex defined earlier for monomial ideals makes sense for monomial modules, too, as does the theorem on free resolutions of generic objects.

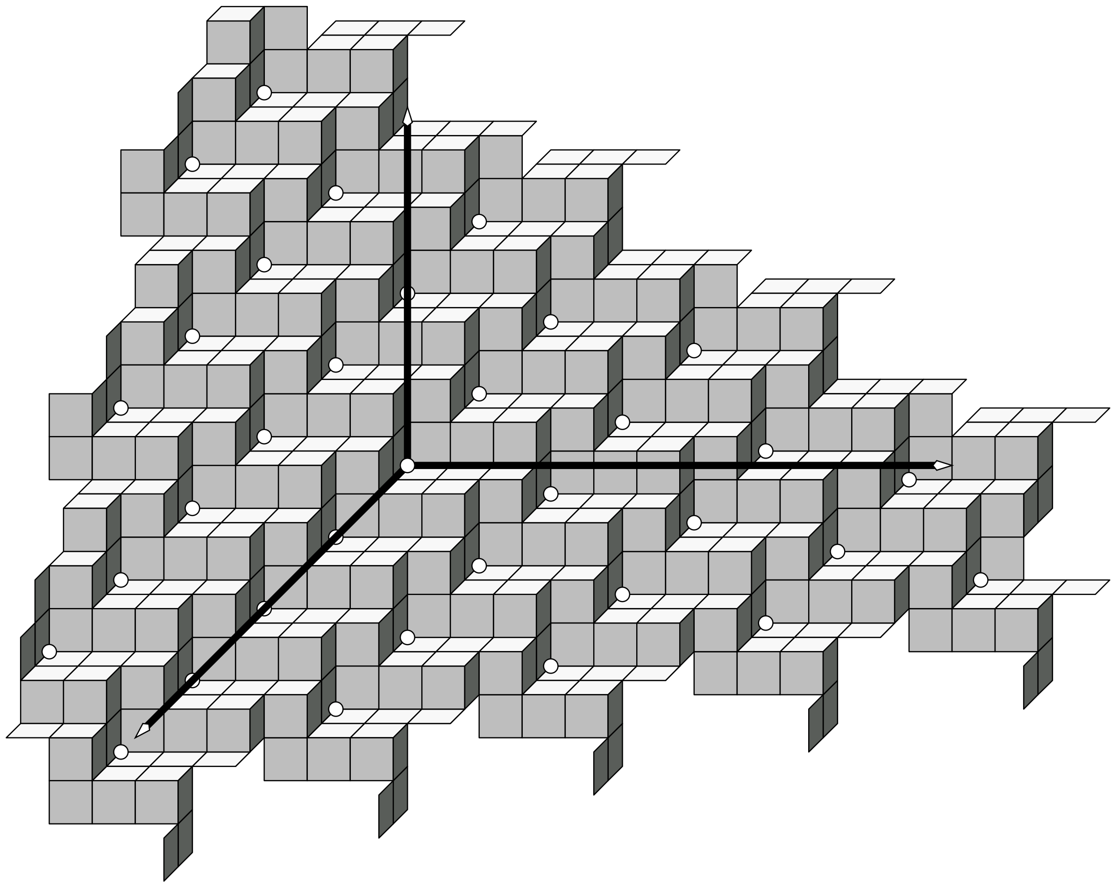
Theorem 7.9 *For a generic monomial module M, the following coincide:*

1. *the Scarf complex of M;*
2. *the hull resolution of M; and*
3. *the minimal free resolution of M.*

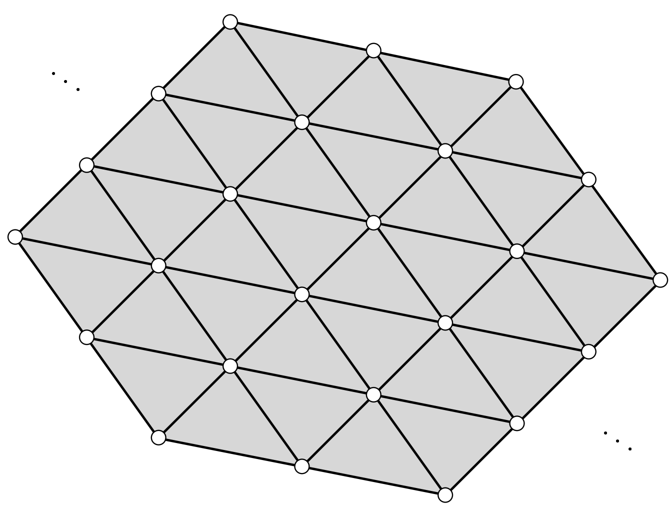
Example 7.10 The lattice $L = \ker([4\ 3\ 5]) \subset \mathbb{Z}^3$ yields a *generic lattice module*

$$M_L = \langle x^i y^j z^k \mid 4i + 3j + 5k = 0 \rangle$$

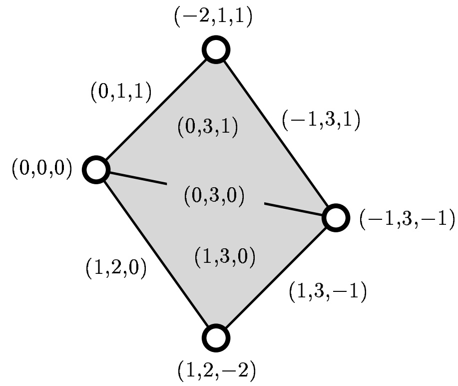
in three variables whose staircase is pictured below.



The hull = Scarf complex is a triangulation of $\mathbb{R} \otimes L$ using L for vertices:



The labeling on every pair of up and down triangles is obtained from the representative labeling



by adding some vector in L to all of the labels. □

7.4 Lattice ideals

For a lattice $L \subset \mathbb{Z}^n$, define the *lattice ideal*

$$I_L = \langle \mathbf{x}^\mathbf{a} - \mathbf{x}^\mathbf{b} \mid \mathbf{a} - \mathbf{b} \in L \text{ and } \mathbf{a}, \mathbf{b} \in \mathbb{N}^n \rangle \subset S.$$

Assume, from now on, that $L \cap \mathbb{N}^n = \{\mathbf{0}\}$, so that the ideal is homogeneous with respect to some grading where $\deg(x_i)$ is a positive integer (use a positive functional which vanishes on L to define the $\mathbb{Z}$-grading). Such ideals arise in a wide variety of contexts, from integer programming and toric geometry to mirror symmetry and hypergeometric differential equations (see [33] and [31]). Given L, the fundamental things we would like to know about I_L are:

1. generators for I_L;
2. the $\mathbb{Z}^n/L$-*graded* Hilbert series of S/I_L, as a rational function; and
3. a (minimal) free resolution of S/I_L over S.

Of course, $3 \Rightarrow 2 \Rightarrow 1$, so we'll aim for free resolutions.

The essential idea is to express the lattice ideal I_L as a quotient of the monomial module M_L by the action of L. In order to do that, let's formalize the action by introducing the *group algebra* $S[L]$ of L over S. Explicitly, this is the algebra

$$S[L] = k[\mathbf{x}^\mathbf{a}\mathbf{z}^\mathbf{b} \mid \mathbf{a} \in \mathbb{N}^n \text{ and } \mathbf{b} \in L] \subset S[z_1^{\pm 1}, \ldots, z_n^{\pm 1}],$$

which carries a $\mathbb{Z}^n$-grading via $\deg(\mathbf{x}^\mathbf{a}\mathbf{z}^\mathbf{b}) = \mathbf{a} + \mathbf{b}$. The action of L on the lattice module $M_L = \langle \mathbf{x}^\mathbf{a} \mid \mathbf{a} \in L \rangle \subset T$ defined in the previous lecture can be expressed as $\mathbf{b} \cdot \mathbf{x}^\mathbf{a} = \mathbf{x}^{\mathbf{a}+\mathbf{b}}$ for $\mathbf{b} \in L$ and $\mathbf{x}^\mathbf{a} \in M_L$. This action is reformulated using the group algebra by stipulating that $\mathbf{x}^\mathbf{a}\mathbf{z}^\mathbf{b} = \mathbf{x}^{\mathbf{a}+\mathbf{b}}$, making M_L into a $\mathbb{Z}^n$-graded module

$$M_L \cong S[L]/\langle \mathbf{x}^\mathbf{a} - \mathbf{x}^\mathbf{b}\mathbf{z}^{\mathbf{a}-\mathbf{b}} \mid \mathbf{a} - \mathbf{b} \in L \text{ and } \mathbf{a}, \mathbf{b} \in \mathbb{N}^n \rangle$$

over $S[L]$, with the image of $\mathbf{x}^\mathbf{a}$ having $\mathbb{Z}^n$-graded degree $\mathbf{a}$, as usual. In fact, any $\mathbb{Z}^n$-graded S-module M that has an equivariant action of L (i.e. commuting with the action of S) such that $\mathbf{b} \in L$ acts as a homomorphism of degree $\mathbf{b}$ is naturally a module over $S[L]$. This is really quite a general statement. To be precise, the category of L-equivariant $\mathbb{Z}^n$-graded S-modules is isomorphic to the category

$$\mathcal{A} = \{\mathbb{Z}^n\text{-graded } S[L]\text{-modules}\}\,.$$

How do we define the quotient of an L-equivariant module = $\mathbb{Z}^n$-graded $S[L]$-module M by the action of L? We would like to identify $m \in M$ with $\mathbf{z}^{\mathbf{b}} \cdot m$ whenever $\mathbf{b} \in L$, so that the quotient is an S-module whose elements are orbits of the action of L on M_L. When $M = S[L]$ itself, this quotient is

$$\begin{aligned} S[L]/L &= S[L]/\langle \mathbf{x}^{\mathbf{a}}\mathbf{z}^{\mathbf{b}} - \mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{N}^n \text{ and } \mathbf{b} \in L\rangle \\ &= S[L]/\langle \mathbf{z}^{\mathbf{b}} - 1 \mid \mathbf{b} \in L\rangle \\ &\cong S\,. \end{aligned}$$

But this copy of S is no longer $\mathbb{Z}^n$-graded, because $\mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{a}}\mathbf{z}^{\mathbf{b}}$, which have different $\mathbb{Z}^n$-graded degrees $\mathbf{a}$ and $\mathbf{a}+\mathbf{b}$, map to the same element $\mathbf{x}^{\mathbf{a}}$. On the other hand, all of the preimages in $S[L]$ of $\mathbf{x}^{\mathbf{a}} \in S$ have $\mathbb{Z}^n$-graded degrees which are congruent modulo L, so this copy of S is $\mathbb{Z}^n/L$-*graded*, with $\mathbf{x}^{\mathbf{a}}$ having $\mathbb{Z}^n/L$-graded degree $\mathbf{a} \pmod{L}$.

For more general $\mathbb{Z}^n$-graded $S[L]$-modules M, our quotient module "M/L" will similarly be obtained by "setting all elements $\mathbf{z}^{\mathbf{b}}$ for $\mathbf{b} \in L$ equal to 1". Algebraically, this is just tensoring M over $S[L]$ with $S = S[L]/\langle \mathbf{z}^{\mathbf{b}} - 1 \mid \mathbf{b} \in L\rangle$,

$$M/L = M \otimes_{S[L]} S[L]/L = M \otimes_{S[L]} S\,.$$

As with $S[L]/L$, the quotient M/L is no longer $\mathbb{Z}^n$-graded, but only $\mathbb{Z}^n/L$-graded. This tensor product therefore defines a functor

$$\pi : \mathcal{A} \longrightarrow \mathcal{B} = \{\mathbb{Z}^n/L\text{-graded } S\text{-modules}\}$$

of categories. Our motivating example is M_L, whose image in $\mathcal{B}$ is obtained from its presentation as a quotient of $S[L]$ given above by setting all occurrences of $\mathbf{z}$ to 1:

$$\pi(M_L) = S[L]/\langle \mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}}\mathbf{z}^{\mathbf{a}-\mathbf{b}} \mid \mathbf{a} - \mathbf{b} \in L \text{ and } \mathbf{a}, \mathbf{b} \in \mathbb{N}^n\rangle|_{\mathbf{z}=1} = S/I_L\,.$$

Therefore we have achieved our goal of writing S/I_L as the quotient of M_L by the action of L. The great thing about π is that it doesn't forget anything significant.

Theorem 7.11 *The functor $\pi : \mathcal{A} \to \mathcal{B}$ sending $M \mapsto M/L$ is an equivalence of categories.*

The main idea of the proof is that if L acts equivariantly on a $\mathbb{Z}^n$-graded S-module in such a way that $\mathbf{b} \in L$ acts as a homomorphism of degree $\mathbf{b}$, then the action of L is necessarily free—no nonzero $\mathbf{b} \in L$ can have any nonzero fixed points. Therefore, an "inverse" to π is given by taking the universal cover.

Now we can use the functoriality of π to do for free resolutions what it did for modules.

Corollary 7.12 *If $\mathbb{F}$ is a free resolution of M_L over $S[L]$, then $\pi(\mathbb{F})$ is a $\mathbb{Z}^n/L$-graded resolution of S/I_L over S.*

And what is a resolution of M_L over $S[L]$? It's just a resolution of M_L as an S-module along with a free action of L. These exist because π is an equivalence. But we have already noted that L acts on nice resolutions of M_L such as the hull resolution.

Definition 7.13 The *hull resolution* of S/I_L is $\pi(\mathbb{F}_{\mathrm{hull}(M_L)})$.

Theorem 7.14 *The hull resolution of S/I_L is a finite free resolution, of length $\leq n$.*

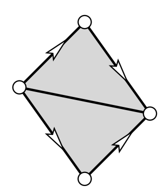
Proof: L acts freely on hull(M_L), which implies that $\mathbb{F}_{\mathrm{hull}(M)}$ is a free $S[L]$-module. Since π(free $S[L]$-module) is a free S-module, the hull resolution of S/I_L is a resolution by free S-modules. The finiteness is because of the local finiteness of Proposition 7.8. And the length is $\leq n$ because hull(M) is a polyhedron inside $\mathbb{R}^n$. □

The rest of this section (and lecture) is devoted to examples.

Example 7.15 The hull resolution of a generic lattice ideal is π(Scarf complex). For instance, the ideal for the image of the affine monomial curve $t \mapsto (t^4, t^3, t^5)$ is the generic lattice ideal

$$I_L = \langle xy^2 - z^2, xz - y^3, yz - x^2 \rangle ,$$

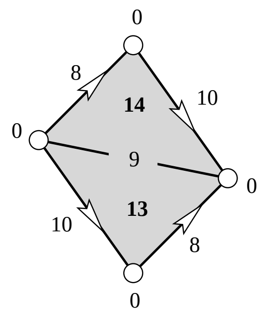
where L is the kernel of the matrix $[4\ 3\ 5]$ over $\mathbb{Z}$. The corresponding lattice module is the one whose staircase and Scarf complex are drawn in Example 7.10. Modulo the lattice, we see that the resolution of S/I_L is a torus



whose fundamental domain is labeled with vectors in Example 7.10. (So π really is a universal cover, as suggested by the proof of Theorem 7.11.) Under the functor π, the resolution of S/I_L is

$$0 \longleftarrow S \xleftarrow{\begin{pmatrix} xy^2-z^2 & xz-y^3 & yz-x^2 \end{pmatrix}} S^3 \xleftarrow{\begin{pmatrix} y & x \\ x & z \\ z & y^2 \end{pmatrix}} S^2 \longleftarrow 0 .$$

To get the $\mathbb{Z}^3/L \cong \mathbb{Z}$-graded Hilbert series, we simply take the alternating sum of the $\mathbb{Z}^3/L$-degrees of the faces of hull$(M_L)/L$ and divide by the appropriate denominator. Each $\mathbb{Z}^3/L$-degree is the dot product of the label in Example 7.10 with $(4, 3, 5)$:



The graded Euler characteristic gives the numerator of the Hilbert series

$$\frac{1 - t^8 - t^9 - t^{10} + t^{13} + t^{14}}{(1-t^4)(1-t^3)(1-t^5)} = \frac{1}{1-t} - t - t^2 = \sum\{t^i \mid i \in \mathbb{N}\{4,3,5\}\}$$

of S/I_L, where $\mathbb{N}\{4,3,5\}$ is the submonoid of $\mathbb{N}$ generated by $\{4,3,5\}$. The denominator comes from the $\mathbb{Z}^3/L$-graded Hilbert series $\frac{1}{(1-t^4)(1-t^3)(1-t^5)}$ of $k[x,y,z]$. □

Example 7.16 Things become much more complicated in four dimensions. The smallest codimension 1 generic lattice module in 4 variables is determined by the lattice $L = \ker([20\ 24\ 25\ 31]) \subset \mathbb{Z}^4$. The lattice ideal I_L is the ideal of the monomial curve $t \mapsto (t^{20}, t^{24}, t^{25}, t^{31})$ in affine four-space. The group algebra is $S[L] = k[a,b,c,d][\mathbf{z}^{\mathbf{a}} \mid \mathbf{a} \in L]$, and

$$\begin{aligned} M_L \;=\; & S[L]/\langle a^4 - bcd\,\mathbf{z}^*, a^3c^2 - b^2d^2\,\mathbf{z}^*, a^2b^3 - c^2d^2\,\mathbf{z}^*, ab^2c - d^3\,\mathbf{z}^*, \\ & b^4 - a^2cd\,\mathbf{z}^*, b^3c^2 - a^3d^2\,\mathbf{z}^*, c^3 - abd\,\mathbf{z}^*\rangle\,, \end{aligned}$$

where, for instance the * in $a^4 - bcd\,\mathbf{z}^*$ is the vector in L which is 4 times the first generator minus 1 times each of the second, third, and fourth generators. The hull = Scarf = minimal resolution of S/I_L is

$$0 \longleftarrow S \longleftarrow S^7 \longleftarrow S^{12} \longleftarrow S^6 \longleftarrow 0\,.$$

Up to the action of L, there are 6 tetrahedra corresponding to the second syzygies and 12 triangles corresponding to the first syzygies. □

Example 7.17 Suppose that L is unimodular, i.e. for all $\sigma \subseteq \{1,\ldots,n\}$ the coordinate projection $\mathbb{Z}^n/(L + \sum_{i\in\sigma}\mathbb{Z}\mathbf{e}_i)$ of $\mathbb{Z}^n/L$ is torsion free. Consider the Lawrence lifting, $\Lambda(L) = \{(\mathbf{u}, -\mathbf{u}) \in \mathbb{Z}^{2n} \mid \mathbf{u} \in L\}$, and its corresponding lattice ideal

$$I_{\Lambda(L)} = \langle \mathbf{x^a y^b} - \mathbf{x^b y^a} \mid \mathbf{a} - \mathbf{b} \in L\rangle \subset k[x_1,\ldots,x_n,y_1,\ldots,y_n]\,.$$

These *unimodular Lawrence ideals* have a remarkable rigidity property: all of their initial ideals are squarefree. In fact, this condition is equivalent to unimodularity for Lawrence ideals.

The hull resolution of $I_{\Lambda(L)}$ is not necessarily minimal, even if L is unimodular. However, the minimal resolution does come from a cellular resolution of $M_{\Lambda(L)}$, and is described by the following combinatorial construction.

Step 1. Take the infinite hyperplane arrangement $\{x_i = j \mid i = 1, \ldots, n,\ j \in \mathbb{Z}\}$.

Step 2. Let $\mathcal{H}_L$ be its intersection with $L \otimes \mathbb{R}$.

Step 3. Form the quotient $\mathcal{H}_L/L$.

In particular, the i-faces of $\mathcal{H}_L/L$ are in 1-1 correspondence with the minimal i-th syzygies of $I_{\Lambda(L)}$. A particular example of the resolution described here is the Eagon-Northcott complex for the 2×2 minors of a generic $2 \times n$ matrix [3]. □

8 Lecture VIII: Local cohomology

Among the first homological objects to be calculated explicitly for squarefree monomial ideals I were the local cohomology modules of S/I with support on $\mathfrak{m}$. Their Hilbert series were described in an unpublished theorem of M. Hochster (which finally found its way into [32]), and their module structure was later described explicitly by H. G. Gräbe [17]. On the other hand, local cohomology with support on the irrelevant monomial ideal in the Cox homogeneous coordinate ring can be used to compute sheaf cohomology on toric varieties, in analogy to the case of projective space [13, 28]. This application has prompted research into local cohomology with support on monomial ideals in general. N. Terai [38] made the calculation of the Hilbert series of the local cohomology modules of S with support on I, inspired by the techniques of Hochster; at the same time, M. Mustaţă [29] found the module structure. Then, using generalizations of Alexander duality for modules, E. Miller [24] showed that the answers provided by Terai and Mustaţă are actually equivalent to those provided by Hochster and Gräbe. Our aim in this final lecture is to present these results, along with the necessary background, in the spirit of geometry from previous lectures.

8.1 Preliminaries

The Čech complex. Local cohomology has many equivalent definitions, each with its own role in the theory. Hochster's local cohomology formula is proved using the most accessible of these, the *Čech complex* (more accurately called the *stable Koszul complex*),

$$\mathbb{C}^{\bullet} : 0 \to S \to \bigoplus_{i=1}^{n} S[x_i^{-1}] \to \cdots \to \bigoplus_{|\sigma|=r} S[x^{-\sigma}] \to \cdots \to S[x_1^{-1}, \ldots, x_n^{-1}] \to 0$$

(see Section 0.1 for notation). This is to be considered as a cochain complex (upper indices increasing from the copy of S sitting in cohomological degree 0), with the map between the summands $S[\mathbf{x}^{-\sigma}] \to S[\mathbf{x}^{-(\sigma \cup i)}]$ in $\mathbb{C}^{\bullet}$ being $\operatorname{sign}(i, \sigma \cup i)$ times the canonical localization homomorphism.

One way to think about the Čech complex in a $\mathbb{Z}^n$-graded way is to replace each summand $S[-\sigma]$ in the Koszul complex $\mathbb{K}^{\bullet}$ by the localization $S[\mathbf{x}^{-\sigma}]$. That this replacement is possible can be seen from monomial matrices, whose scalar entries can be used just as easily as maps between localizations as between free modules; in fact this was the original motivation for defining monomial matrices in [24]. To write one down, we employ the symbol $*$ in an exponent vector to indicate that the variable with that index has been inverted. Thus, if $S = k[x, y, z]$, the "shift" $S[(0, *, 0)]$ represents the localization $S[y^{-1}]$, while $S[(0, 0, 0)]$ retains its usual meaning (no variables are to be inverted).

Example 8.1 The Čech complex in 3 variables looks like

$$0 \longrightarrow S \xrightarrow{\begin{array}{c|ccc} & *00 & 0{*}0 & 00* \\ 000 & 1 & 1 & 1 \end{array}} \begin{array}{c} S[x^{-1}] \\ \oplus \\ S[y^{-1}] \\ \oplus \\ S[z^{-1}] \end{array} \xrightarrow{\begin{array}{c|ccc} & 0{*}{*} & {*}0{*} & {*}{*}0 \\ *00 & 0 & 1 & 1 \\ 0{*}0 & 1 & 0 & -1 \\ 00* & -1 & -1 & 0 \end{array}} \begin{array}{c} S[(yz)^{-1}] \\ \oplus \\ S[(xz)^{-1}] \\ \oplus \\ S[(xy)^{-1}] \end{array} \xrightarrow{\begin{array}{c|c} & {*}{*}{*} \\ 0{*}{*} & 1 \\ {*}0{*} & -1 \\ {*}{*}0 & 1 \end{array}} S[(xyz)^{-1}] \longrightarrow 0\,,$$

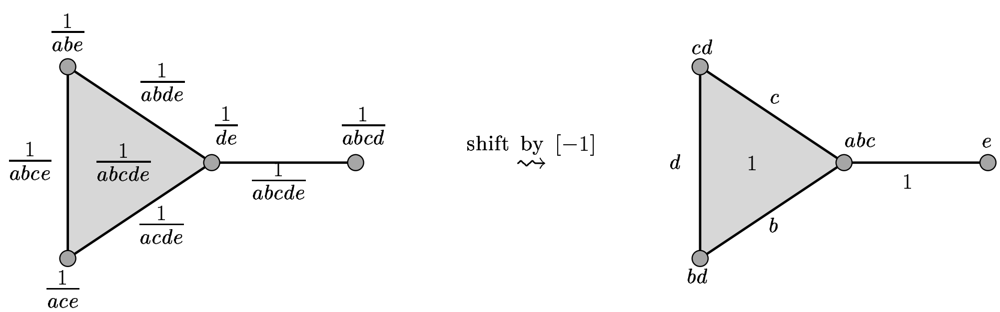
where the vectors are row vectors and the matrices act by multiplication on the right (as is natural for the cochain complex—see Example 0.6). □

Definition 8.2 Given a $\mathbb{Z}^n$-graded module M, the r^{th} *local cohomology module of* M *with support on* $\mathfrak{m}$ is the r^{th} cohomology $H^r_{\mathfrak{m}}(M) = H^r(M \otimes_S \check{C}^{\bullet})$.

The local cohomology of a finitely generated module M with support on $\mathfrak{m}$ is related to other homological invariants of M, via *local duality.* Before stating this fundamental theorem, we need to say something about Ext and Matlis duality.

Ext modules. Let $0 \leftarrow M \leftarrow \mathbb{F}_{\bullet}$ be a free resolution of M. Then $\mathrm{Hom}_S(\mathbb{F}_{\bullet}, S)$ is a cochain complex of free S-modules whose r^{th} cohomology is $\mathrm{Ext}^r_S(M, S)$, by definition. Recall that $\mathrm{Hom}_S(S[-\mathbf{a}], S) = S[\mathbf{a}]$; thus, in terms of monomial matrices, $\mathrm{Hom}_S(\mathbb{F}_{\bullet}, S)$ is obtained from $\mathbb{F}_{\bullet}$ by taking the negatives of the vector labels (inverses of monomial labels) on the columns and rows while taking the transposes of the scalar matrices.

Example 8.3 If $M = S/I$ has a cellular resolution supported on the labeled cell complex X, then the complex $\mathrm{Hom}_S(\mathbb{F}_X, S)$ is obtained by using as scalars the cochain complex of X, with the inverses of the monomial labels. For instance, if $I = I_\Gamma$ from Example 1.4 which has a cellular resolution $\mathbb{F}_X$ given in Example 1.5, then $\mathbb{F}^X := \mathrm{Hom}_S(\mathbb{F}_X, S)$ is obtained from surrounding the cochain complex of X in a monomial matrix with the labels at left below.



For various reasons, it is common to work with the $\mathbb{Z}^n$-graded shift $\mathrm{Hom}_S(\mathbb{F}_\cdot, S)[-\mathbf{1}] = \mathrm{Hom}_S(\mathbb{F}_\cdot, S[-\mathbf{1}])$. It is especially convenient for the case of $M = S/I$ for squarefree I, because the vector labels in monomial matrices for this shift are again in $\mathbb{N}^n$ (i.e. the monomials are again in S), as in the right picture above. The module $S[-\mathbf{1}]$ is sometimes called the *canonical module of* S, and denoted by ω_S. □

Matlis duality. A $\mathbb{Z}^n$-graded module M is really just a collection of vector spaces $M_\mathbf{a}$ indexed by $\mathbf{a} \in \mathbb{Z}^n$, along with a given set of maps $\lambda_{\mathbf{a}\to\mathbf{b}} := \cdot\mathbf{x}^{\mathbf{b}-\mathbf{a}} : M_\mathbf{a} \to M_\mathbf{b}$ for each pair $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$ with $\mathbf{a} \preceq \mathbf{b}$. Given M we can make another such collection of vector spaces and homomorphisms from the k-duals $\mathrm{Hom}_k(M_\mathbf{a}, k)$ and the transposes $\lambda^{\mathbf{a}\leftarrow\mathbf{b}} : \mathrm{Hom}_k(M_\mathbf{b}, k) \to \mathrm{Hom}_k(M_\mathbf{a}, k)$, as follows.

Definition 8.4 The *Matlis dual of* M is the module M^* whose $\mathbb{Z}^n$-graded piece in degree $-\mathbf{a}$ is $M^*_{-\mathbf{a}} = \mathrm{Hom}_k(M_\mathbf{a}, k)$. For $\mathbf{a} \preceq \mathbf{b} \in \mathbb{Z}^n$, the action of multiplication $\cdot\mathbf{x}^{\mathbf{b}-\mathbf{a}} : M^*_{-\mathbf{b}} \to M^*_{-\mathbf{a}}$ is by $\lambda^{\mathbf{a}\leftarrow\mathbf{b}}$.

From the point of view of Hilbert series, Matlis duality simply reverses the grading.

Lemma 8.5 $H(M^*; x_1, \ldots, x_n) = H(M; x_1^{-1}, \ldots, x_n^{-1})$.

If $M \to N$ is a homomorphism of $\mathbb{Z}^n$-graded modules, then Matlis duality induces a dual map $M^* \leftarrow N^*$. That is, Matlis duality is a contravariant functor. In addition, Matils duality is *exact*, since in each degree it is the functor $\mathrm{Hom}_k(-, k)$, which is exact. We are justified in calling this functor a duality, since a k-vector space is canonically isomorphic to its double-dual, at least when it is finite-dimensional.

Remark 8.6 For those that like the language of categories, a $\mathbb{Z}^n$-graded module M whose $\mathbb{Z}^n$-graded degrees are finite-dimensional is a subcategory of the category $\mathcal{K}$ of finite-dimensional k-vector spaces, with the morphisms in M being determined by monomials in S. Matlis duality is an anti-isomorphism of M with another subcategory M^* of $\mathcal{K}$ which is induced by the usual anti-isomorphism $\mathcal{K} \to \mathcal{K}$ given by the transpose $\mathrm{Hom}_k(-, k)$. □

Now we are ready to state the local duality theorem, equating local cohomology with support on $\mathfrak{m}$ with the Matlis dual of an Ext module. For references and a general proof, see [7]; for a special treatment of the $\mathbb{Z}^n$-graded case, see [16] or [24].

Theorem 8.7 $H^r_\mathfrak{m}(M) \cong \mathrm{Ext}^{n-r}_S(M, \omega_S)^*$ *if* M *is finitely generated and* $\mathbb{Z}^n$*-graded.*

8.2 Maximal support

The theorem of Hochster [32, Theorem II.4.1] gives the $\mathbb{Z}^n$-graded Hilbert series $H(H^r_\mathfrak{m}(S/I_\Delta); \mathbf{x})$ for the local cohomology of S/I_Δ with support on $\mathfrak{m}$ in terms of the homology of the links in the Stanley-Reisner complex Δ.

Theorem 8.8 $H(H^r_\mathfrak{m}(S/I_\Delta); \mathbf{x}) = \displaystyle\sum_{\sigma\in\Delta} \dim_k \widetilde{H}^{r-|\sigma|-1}(\mathrm{link}_\sigma \Delta; k) \prod_{j\in\sigma} \frac{x_j^{-1}}{1 - x_j^{-1}}$.

Let us parse the statement. The product $\prod_{j\in\sigma}\frac{x_j^{-1}}{1-x_j^{-1}}$ is the sum of all Laurent monomials whose exponent vectors are nonpositive and have support exactly σ. Therefore, the formula for the Hilbert series of $H^r_{\mathfrak{m}}(S/I_\Delta)$ is just like the one for S/I_Δ in the third line of the displayed equation in Section 1.2, except that we are considering monomials with negative exponents, and we have added in nonnegative constants $\dim_k \widetilde{H}^{r-|\sigma|-1}(\mathrm{link}_\sigma\Delta;k)$ depending on r and σ.

As one might expect from the relation between $\mathbb{K}^\bullet$ and $\mathbb{C}^\bullet$, the proof of Theorem 8.8 is quite similar to that of Theorem 1.8, being accomplished (as usual) by checking which simplicial complex has its cochain complex in each $\mathbb{Z}^n$-graded degree. The main complication is in determining what relation the module $S/I_\Delta\otimes S[\mathbf{x}^{-\sigma}]$ has to the face ring of something (it turns out that the "something" is a link in Δ). The reader wishing details should consult [7] or [32].

From Hochster's Hilbert series formula, we find that whenever $b_i<-1$, the vector spaces $H^r_{\mathfrak{m}}(S/I_\Delta)_{\mathbf{b}}$ and $H^r_{\mathfrak{m}}(S/I_\Delta)_{\mathbf{b}+\mathbf{e}_i}$ have the same dimension. It is natural, therefore, to think that multiplication by x_i is an isomorphism between these graded components. This is indeed the case, as one can check even for the complex $S/I_\Delta\otimes\mathbb{C}^\bullet$ whose homology is $H^r_{\mathfrak{m}}(S/I_\Delta)$. But what about when $b_i=-1$? The answer is provided by an observation of H. G. Gräbe [17].

Theorem 8.9 *Identify each graded piece* $H^r_{\mathfrak{m}}(S/I_\Delta)_{\mathbf{b}}$ *with* $\widetilde{H}^{r-|\sigma|-1}(\mathrm{link}_\sigma\Delta;k)$ *whenever* $\mathbf{b}\preceq\mathbf{0}$ *and* $\mathrm{supp}(\mathbf{b})=\sigma$. *Given* $\sigma\in\Delta$ *and* $i\notin\sigma$ *there are maps*

$$\widetilde{H}^{r-|\sigma\cup i|-1}(\mathrm{link}_{\sigma\cup i}\Delta;k)\longrightarrow\widetilde{H}^{r-|\sigma|-1}(\mathrm{link}_\sigma\Delta;k)$$

which agree with multiplication by x_i *on the graded piece* $H^r_{\mathfrak{m}}(S/I_\Delta)_{\mathbf{b}}$ *as above whenever* $b_i=-1$.

The choice of cohomology over homology in the Theorem may seem to be an error, given that the inclusion $\mathrm{link}_{\sigma\cup i}\Delta\to\mathrm{link}_\sigma\Delta$ of simplicial complexes induces maps on cohomology the other way. However, Gräbe's maps are not the usual ones induced by the inclusion; instead, they arise because the *co*chain complex $\widetilde{\mathcal{C}}^\bullet(\mathrm{link}_{\sigma\cup i}\Delta;k)$ can be made into a *sub*complex of $\widetilde{\mathcal{C}}^\bullet(\mathrm{link}_\sigma\Delta;k)$ by sending $e^*_\tau\mapsto\mathrm{sign}(i,\sigma\cup i)e^*_{\tau\cup i}$. The geometric content of such maps is not so easy to describe directly, but see Corollary 8.21, where they are related to geometry in $\Delta^\vee$ via Theorem 8.12, below.

8.3 Monomial support

An element m in a module M is said to have *support on an ideal* I if m is annihilated by some power of I. The reason why $H^r_{\mathfrak{m}}(-)$ is said to have *support on* $\mathfrak{m}$ is because $x_1,\ldots,x_n$ generate $\mathfrak{m}$, so the kernel of the map $M\otimes S\to\bigoplus_i M\otimes S[x_i^{-1}]$ is the set of elements which are annihilated by some power of each x_i—that is, those elements with support on $\mathfrak{m}$. If, in the Čech complex on s variables, we replace each variable x_i by a monomial m_i, then we get a complex $\mathbb{C}^\bullet(m_1,\ldots,m_s)$ whose first map is $S\to\bigoplus_i S[m_i^{-1}]$. Tensoring $\mathbb{C}^\bullet(m_1,\ldots,m_s)$ with M, the kernel of this first map becomes the set of elements in M which have support on $I=\langle m_1,\ldots,m_s\rangle$.

Definition 8.10 The r^{th} *local cohomology of a module M with support on $I = \langle m_1, \ldots, m_s \rangle$* is the r^{th} cohomology $H_I^r(M)$ of the complex $M \otimes_S \mathbb{C}^{\bullet}(m_1, \ldots, m_s)$.

Again, there are many equivalent ways to define local cohomology with support on I, with the Čech complex being perhaps the most straightforward. One of the others characterizes $H_I^{\bullet}(M)$ as the right derived functors of $H_I^0(M) =$ "taking the set of elements with support on I". The standard way to calculate such derived functors is to take an injective resolution of M, apply $H_I^0(-)$ to it, and set $H_I^r(M)$ to be the r^{th} cohomology of the resulting complex. (For $\mathbb{Z}^n$-graded M and I, a $\mathbb{Z}^n$-graded injective resolution of M suffices.) But because I is finitely generated, $H_I^0(M)$ depends only on the radical of I, so we can assume I is a radical ideal whenever we write $H_I^{\bullet}(-)$. In this lecture, that means I is a squarefree monomial ideal. We will mention more of the equivalent characterizations of $H_I^{\bullet}(-)$ below; in particular, one of the purposes of Section 8.4 following this one is to show which other complexes can replace $\mathbb{C}^{\bullet}(m_1, \ldots, m_s)$ in the definition.

Inspired by Theorem 8.8 and its proof, N. Terai calculated the following [38].

Theorem 8.11 *The $\mathbb{Z}^n$-graded Hilbert series of $H_{I_\Delta}^r(S)$ is*

$$H(H_{I_\Delta}^r(S); \mathbf{x}) = \sum_{\sigma \in \Delta} \dim_k \widetilde{H}_{n-r-|\sigma|-1}(\operatorname{link}_\sigma \Delta; k) \prod_{i \in \overline{\sigma}} \frac{x_i^{-1}}{1 - x_i^{-1}} \prod_{j \in \sigma} \frac{1}{1 - x_j}.$$

This series is interpreted in a manner similar to Hochster's: $\prod_{i \in \overline{\sigma}} \frac{x_i^{-1}}{1-x_i^{-1}} \prod_{j \in \sigma} \frac{1}{1-x_j}$ is the sum of all Laurent monomials $\mathbf{x}^{\mathbf{b}}$ whose negative parts $\mathbf{b}_- := \sum_{b_i < 0} -b_i \mathbf{e}_i$ have support precisely $\overline{\sigma}$. Terai's proof follows the outline of Hochster's, using $\mathbb{C}^{\bullet}(m_1, \ldots, m_s)$ in place of $\mathbb{C}^{\bullet}$. Terai's argument is somewhat more complicated, though, since the simplicial complexes in his formula have n vertices, while those obtained from the Čech complex have as many vertices as there are generators of I_Δ.

Again, it is clear from Theorem 8.11 that many of the multiplication maps $\cdot x_i$ are between vector spaces of the same dimension, and as before, one can verify using the Čech complex of I_Δ that these maps are isomorphisms. While Terai made his calculation, M. Mustaţă simultaneously and independently observed these isomorphisms and went further, giving the module structure in terms of simplicial cohomology [29].

Theorem 8.12 *Each graded piece $H_{I_\Delta}^r(S)_{\mathbf{b}}$ is isomorphic to $\widetilde{H}^{r-2}(\Delta^\vee|_{\overline{\sigma}}; k)$, where $\mathbf{b} \in \mathbb{Z}^n$ and $\operatorname{supp}(\mathbf{b}_-) = \overline{\sigma}$. For $i \in \overline{\sigma}$, the maps on simplicial cohomology*

$$\widetilde{H}^{r-2}(\Delta^\vee|_{\overline{\sigma}}; k) \longrightarrow \widetilde{H}^{r-2}(\Delta^\vee|_{\overline{\sigma}\setminus i}; k)$$

induced by the inclusions $\Delta^\vee|_{\overline{\sigma}\setminus i} \to \Delta^\vee|_{\overline{\sigma}}$ agree with multiplication by x_i on the graded piece $H_{\mathfrak{m}}^r(S/I_\Delta)_{\mathbf{b}}$ whenever $b_i = -1$. If $b_i \neq -1$, then multiplication by x_i is an isomorphism on $H_{\mathfrak{m}}^r(S/I_\Delta)_{\mathbf{b}}$.

Upon comparison of Mustaţă's theorem with Terai's, it shouldn't be too surprising that the graded pieces of $H_{I_\Delta}^r(S)$ can be identified with $\widetilde{H}^{r-2}(\Delta^\vee|_{\overline{\sigma}}; k)$; after all,

$$\widetilde{H}_{n-r-|\sigma|-1}(\operatorname{link}_\sigma \Delta; k) \cong \widetilde{H}^{r-2}(\Delta^\vee|_{\overline{\sigma}}; k)$$

by Alexander duality in the form of Corollary 6.6. We will see in the next section why the theorems in this section look so similar to those in the previous section, and why duality enters the picture.

There is another proof of the theorems in this section which exploits *cellular injective resolutions*, particularly that of the canonical module ω_S, whose minimal $\mathbb{Z}^n$-graded injective resolution is a simplex (and Matlis dual to the usual Čech complex $\mathbb{C}^\bullet$). The point is that applying $H^0_{I_\Delta}(-)$ directly to the minimal $\mathbb{Z}^n$-graded injective resolution of $S = \omega_S[\mathbf{1}]$ yields a *cellular complex of injectives* whose homology is automatically given in each degree as the simplicial cohomology of a link in Δ. In fact, this proof is almost the same as that of Theorem 1.8; see [27] for details.

8.4 The Čech hull

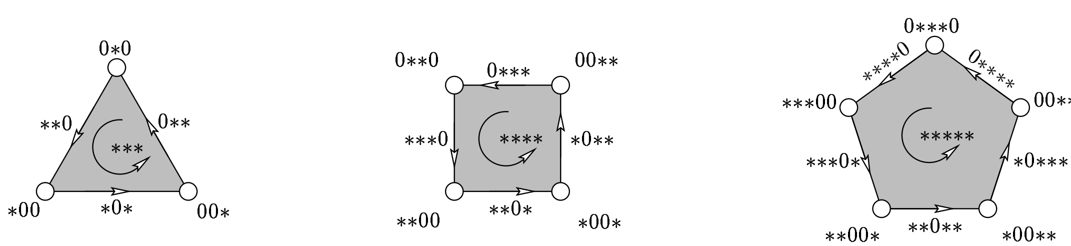
Mustaţă's proof of Theorem 8.12 used cellular resolutions in a particularly nice way, in combination with the characterization of $H^\bullet_I(M)$ as a limit over $t \in \mathbb{N}$ of modules $\mathrm{Ext}^\bullet_S(S/I^t, M)$. He was able to use the Taylor resolution (Section 5.3) to get very nice finite approximations to the limit. Although we will bypass these limits here, let's see what the Taylor resolution of S/I has to do with local cohomology supported on I.

Recall that the Taylor complex of the squarefree monomial ideal $I = \langle \mathbf{x}^{\sigma_1}, \ldots, \mathbf{x}^{\sigma_s} \rangle$ is supported on a simplex X whose s vertices are labeled by the minimal generators. The label on a face $G \in X$ is the monomial $m_G = \mathbf{x}^{\sigma_G} := \mathrm{lcm}\,(\mathbf{x}^{\sigma_j}, j \in G)$, and the monomial matrices for the Taylor resolution are filled with the boundary maps of X. On the other hand, the monomial matrices for the Čech complex $\mathbb{C}^\bullet(\mathbf{x}^{\sigma_1}, \ldots, \mathbf{x}^{\sigma_s})$ are filled with the coboundary maps of the *same* simplex X, while the monomial labels m_G are replaced by the vector labels $*\sigma_G$ that have a $*$ in the i^{th} slot for $i \in \sigma_G$ and 0 otherwise (cf. Example 8.1). The key point is that inverting $\mathbf{x}^{\sigma_i}$ for $i \in G$ is the same thing as inverting m_G; i.e. $S[\mathbf{x}^{-\sigma_i} \mid i \in G] = S[\mathbf{x}^{-\sigma_G}]$.

E. Miller showed how this transition from Taylor complex to Čech complex works for all free resolutions of S/I [24]. When the shifts are squarefree, one can use:

Definition 8.13 Suppose that $\mathbb{F}_\bullet$ is a free resolution of S/I_Δ that has monomial matrices $\{\Lambda_r\}$ with squarefree row and column labels. The *generalized Čech complex* $\mathbb{C}^\bullet_{\mathbb{F}}$ is the complex of localizations of S whose monomial matrix Λ^r is obtained from Λ_r by taking the transpose of the scalar entries and replacing each label σ by $*\sigma$. If $\mathbb{F}_\bullet$ is minimal, $\mathbb{C}^\bullet_{\mathbb{F}}$ is called the *canonical Čech complex of* I and denoted $\mathbb{C}^\bullet_I$.

Example 8.14 The canonical Čech complexes for polygons look like



using the cellular minimal resolutions of Example 5.5. In terms of the applications of local cohomology to sheaf cohomology on toric varieties, these diagrams illustrate how the canonical Čech complex reflects the geometry particularly well when the toric variety is projective and smooth (or merely simplicial). □

We have what seems to be a magical way of taking a complex $\mathbb{F}_{\bullet}$ of free modules and producing a complex $\mathbb{C}^{\bullet}_{\mathbb{F}}$ of localizations of S. But as with other kinds of magic, the secrets of this procedure become more transparent when it is broken down into smaller steps. We start by getting the transposes of the scalar matrices to show up, which they do in $\mathbb{F}^{\bullet} := \mathrm{Hom}_S(\mathbb{F}_{\bullet}, \omega_S)$ as in Example 8.3. Observe that when $S/I \leftarrow \mathbb{F}_{\bullet}$ is a free resolution, the complex $\mathbb{F}^{\bullet}$ of free modules has cohomology $\mathrm{Ext}^{\bullet}_S(S/I, \omega_S)$. Now how do we create localizations?

Definition 8.15 The *Čech hull* of an $\mathbb{N}^n$-graded module M is the $\mathbb{Z}^n$-graded module $\check{C}M$ whose degree $\mathbf{b}$ piece is

$$(\check{C}M)_{\mathbf{b}} = M_{\mathbf{b}_+} \quad \text{where} \quad \mathbf{b}_+ = \sum_{b_i \geq 0} b_i \mathbf{e}_i \,.$$

Equivalently,

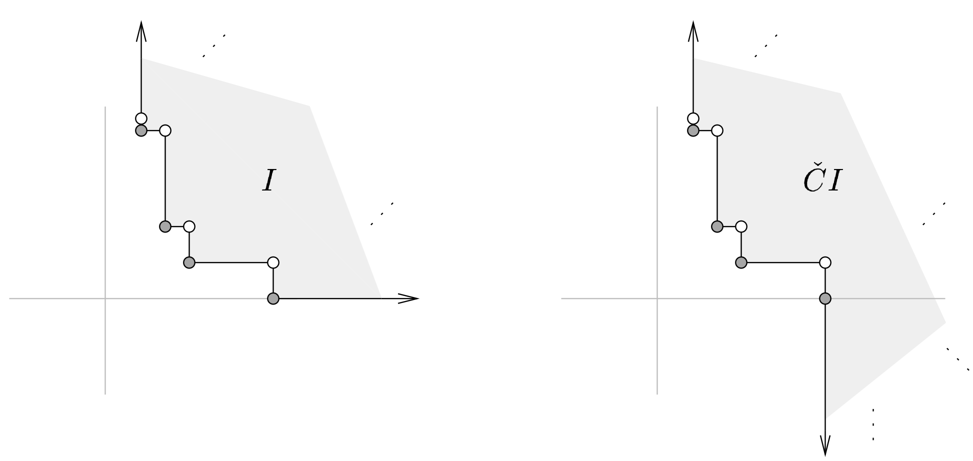
$$\check{C}M = \bigoplus_{\mathbf{b} \in \mathbb{N}^n} M_{\mathbf{b}} \otimes_k k[x_i^{-1} \mid b_i = 0]\,.$$

The action of multiplication by x_i is

$$\cdot x_i : (\check{C}M)_{\mathbf{b}} \to (\check{C}M)_{\mathbf{e}_i + \mathbf{b}} = \begin{cases} \text{identity} & \text{if } b_i < 0 \\ \cdot x_i : M_{\mathbf{b}_+} \to M_{\mathbf{e}_i + \mathbf{b}_+} & \text{if } b_i \geq 0 \end{cases}.$$

Note that $\mathbf{e}_i + \mathbf{b}_+ = (\mathbf{e}_i + \mathbf{b})_+$ whenever $b_i \geq 0$.

This definition was made in [23] for monomial ideals (as part of Alexander duality) where $\check{C}I$ is characterized as the largest monomial module inside of $T = S[x_1^{-1}, \ldots, x_n^{-1}]$ whose intersection with S is precisely I. The staircase diagram of $\check{C}I$ is obtained by pushing to negative infinity any point on the staircase diagram for I which touches the boundary of the positive orthant:



Heuristically, the first description of $\check{C}M$ in the definition says that if you want to know what $\check{C}M$ looks like in degree $\mathbf{b} \in \mathbb{Z}^n$, then check what M looks like in the nonnegative degree closest to $\mathbf{b}$; the second description says that the vector space $M_{\mathbf{a}}$ for $\mathbf{a} \in \mathbb{N}^n$ is copied into all degrees $\mathbf{b}$ such that $\mathbf{b}_+ = \mathbf{a}$.

Getting the desired localizations to appear is easy, using the Čech hull:

Lemma 8.16 $\check{C}(S[-\overline{\sigma}]) = S[\mathbf{x}^{-\sigma}][-\mathbf{1}]$.

If $\lambda : M \to N$ is a map of $\mathbb{N}^n$-graded modules, then there is a map $\check{C}\lambda : \check{C}M \to \check{C}N$ obtained by setting $(\check{C}\lambda)_{\mathbf{b}} = \lambda_{\mathbf{b}_+}$ for $\mathbf{b} \in \mathbb{Z}^n$. In particular, if $\mathbb{F}$ is a complex of free modules generated in squarefree degrees, then $\check{C}\mathbb{F}$ is a complex of $\mathbb{Z}^n$-graded localizations of S, by the Lemma. Recall from Example 8.3 that $\omega_S = S[-\mathbf{1}]$.

Proposition 8.17 *Suppose that* $\mathbb{F}_{\bullet}$ *is a free resolution of* S/I_Δ *whose shifts are all squarefree, and let* $\mathbb{F}^{\bullet} = \operatorname{Hom}_S(\mathbb{F}_{\bullet}, \omega_S)$. *Then* $(\check{C}\mathbb{F}^{\bullet})[\mathbf{1}] = \mathbb{C}^{\bullet}_{\mathbb{F}}$.

Proof: Since taking $\operatorname{Hom}_S(-, \omega_S)$ reverses the arrows (i.e. takes the transposes), the result follows from the Lemma because $\operatorname{Hom}_S(S[-\sigma], \omega_S) = S[-\overline{\sigma}]$. □

The complex $\check{C}\mathbb{F}^{\bullet}$ depends only on $\mathbb{F}_{\bullet}$ and not on the monomial matrix used to represent it; thus the same is true of $\mathbb{C}^{\bullet}_{\mathbb{F}}$ by the Proposition. In particular, the isomorphism class of the canonical Čech complex $\mathbb{C}^{\bullet}_I$, as a complex of $\mathbb{Z}^n$-graded S-modules, depends only on I. Even better, the Proposition can be used to define $\mathbb{C}^{\bullet}_{\mathbb{F}}$ for any free resolution of S/I. Doing so, Miller proved that there are lots of complexes of localizations of S which can take the place of $\mathbb{C}^{\bullet}(\mathbf{x}^{\sigma_1}, \ldots, \mathbf{x}^{\sigma_s})$ in Definition 8.10.

Theorem 8.18 $H^r_{I_\Delta}(M) = H^r(M \otimes \mathbb{C}^{\bullet}_{\mathbb{F}})$ *for any free resolution* $\mathbb{F}_{\bullet}$ *of* S/I_Δ.

If $\mathbb{F}_{\bullet}$ is the Taylor resolution, this Theorem agrees with Definition 8.13. It holds even for $\mathbb{Z}$-graded or ungraded modules M. Taking $M = S$, we see immediately how to calculate $H^r_{I_\Delta}(S)$.

Corollary 8.19 $H^r_{I_\Delta}(S) = (\check{C}\operatorname{Ext}^r_S(S/I_\Delta, \omega_S))[\mathbf{1}]$.

Proof: The Čech hull is *exact*—that is, applying $\check{C}$ to an exact sequence yields an exact sequence—because it copies the degree $\mathbf{b}_+$ part of any complex into degree $\mathbf{b}$. In particular, $\check{C}$ commutes with taking cohomology, as does shifting by $[\mathbf{1}]$. □

Corollary 8.20 *Terai's formula in Theorem 8.11 is equivalent to Hochster's formula in Theorem 8.8.*

Proof: $\operatorname{Ext}^r_S(S/I_\Delta, \omega_S) \cong H^{n-r}_{\mathfrak{m}}(S/I_\Delta)^*$ by Theorem 8.7, so

$$H(\operatorname{Ext}^r_S(S/I_\Delta, \omega_S); \mathbf{x}) = \sum_{\sigma \in \Delta} \dim_k \widetilde{H}_{n-r-|\sigma|-1}(\operatorname{link}_\sigma \Delta; k) \prod_{j \in \sigma} \frac{x_j}{1 - x_j}$$

by Lemma 8.5 applied to Theorem 8.8 (r and x_j^{-1} have been replaced by $n-r$ and x_j). Note that we replace the superscripts on $\widetilde{H}$ by subscripts when we take Matlis duals;

we will see why in the proof of the next corollary. Relying on the second description of the Čech hull in Definition 8.15, this yields

$$H(\check{C}\mathrm{Ext}^r_S(S/I_\Delta, \omega_S); \mathbf{x}) = \sum_{\sigma\in\Delta} \dim_k \widetilde{H}_{n-r-|\sigma|-1}(\mathrm{link}_\sigma \Delta; k) \prod_{j\in\sigma} \frac{x_j}{1-x_j} \prod_{i\in\overline{\sigma}} \frac{1}{1-x_i^{-1}}\,.$$

Shifting the input module by $[\mathbf{1}]$ multiplies this whole expression by $x_1^{-1}\cdots x_n^{-1} = \mathbf{x}^{-\sigma}\mathbf{x}^{-\overline{\sigma}}$, and gives the Hilbert series of $H^r_I(S)$ by Corollary 8.19. All of these steps are reversible, so Hochster's formula can similarly be derived from Terai's. $\square$

Corollary 8.21 *The theorem of Mustaţă (8.12) is equivalent to that of Gräbe (8.9).*

Proof: The proof is the same as the previous corollary, except that we need to keep track of the multiplication maps between the $\mathbb{Z}^n$-graded degrees of the modules in question. The crux is that the homomorphisms

$$\widetilde{H}^{n-r-|\sigma\cup i|-1}(\mathrm{link}_{\sigma\cup i} \Delta; k) \longrightarrow \widetilde{H}^{n-r-|\sigma|-1}(\mathrm{link}_\sigma \Delta; k)$$

coming from Gräbe's theorem become the transpose homomorphisms

$$\widetilde{H}_{n-r-|\sigma|-1}(\mathrm{link}_\sigma \Delta; k) \longrightarrow \widetilde{H}_{n-r-|\sigma\cup i|-1}(\mathrm{link}_{\sigma\cup i} \Delta; k)$$

in the Matlis dual, by definition. Then, by Alexander duality, these agree with the maps

$$\widetilde{H}^{r-2}(\Delta^\vee|_{\overline{\sigma}}; k) \longrightarrow \widetilde{H}^{r-2}(\Delta^\vee|_{\overline{\sigma}\setminus i}; k)$$

in Mustaţă's theorem. $\square$

The Čech hull had been defined before Theorems 8.11 and 8.12 were known, although its relation to local cohomology had not been found. The motivation for Theorem 8.18 was to prove *local duality with monomial support* [24], which generalizes Theorem 8.7—this is where one really needs the fact that the canonical Čech complex has minimal length.

A Appendix: Exercises

The following exercises are derived from those given at the June 1999 COCOA Summer School in Turin, Italy.

A.1 Exercises from Day 1

Exercise A.1.1 Let $n = 6$ and let Δ be the boundary of an octahedron.

(a) Determine I_Δ and $I_{\Delta^\vee}$.
(b) Compute their respective Hilbert series.
(c) Compute their minimal free resolutions.
(d) Interpret the Betti numbers obtained in part (c) in terms of simplicial homology.

Exercise A.1.2 Explain how CoCoA can be used to calculate the homology of a simplicial complex.

Exercise A.1.3 Consider a 5×5-matrix of indeterminates (x_{ij}) and let I_Δ be the ideal in $k[x_{ij}]$ generated by all 100 squarefree monomials of the form $x_{ij}x_{il}$ or $x_{ij}x_{lj}$. The simplicial complex Δ is called the *chessboard complex*.

(a) Using Hilbert series in CoCoA, find the number of 2-dimensional faces of Δ.
(b) Compute the homology groups of Δ, using your algorithm in Exercise A.1.2.

Exercise A.1.4 Let I be the ideal of the cubic Veronese surface in projective 9-space. Compute the generic initial ideal of I for reverse lexicographic order and for purely lexicographic order. How do their minimal free resolutions compare to that of I itself?

Exercise A.1.5 Give an example of a Borel-fixed ideal which is not the initial monomial ideal of any homogeneous prime ideal in $k[x_1, \ldots, x_n]$. Are such examples rare or abundant?

Exercise A.1.6 Let $I \subset \mathbb{C}[x, y, z]$ be the homogeneous radical ideal of seven generic points in $\mathbb{P}^2_{\mathbb{C}}$, where $\mathbb{C}$ is the field of complex numbers. List **all** initial monomial ideals of I, with respect to all term orders.

Exercise A.1.7 Let M be an arbitrary monomial ideal in $\mathbb{C}[x_1, \ldots, x_n]$, and let $\mathcal{B} \subset \mathbb{N}^n$ be the set of all vectors $\mathbf{b}$ such that $\mathbf{x}^{\mathbf{b}}$ is not in M. The *distraction* of M is the radical ideal D_M of all polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ which vanish on the set $\mathcal{B}$.

(a) Determine a finite generating set of D_M.
(b) Show that M is the initial monomial ideal of D_M with respect to any term order.
(c) Determine the prime decomposition of D_M.
(d) The number of prime components of D_M is called the *arithmetic degree* of M. Write a CoCoA program for computing the arithmetic degree.

A.2 Exercises from Day 2

Exercise A.2.1 Let Δ be the simplicial complex on the set $\{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z_1, z_2, z_3, z_4\}$ obtained by **polarization** of the monomial ideal

$$M = \langle x^4, y^4, z^4, x^3y^2z, xy^3z^2, x^2yz^3 \rangle.$$

Determine the number of i-dimensional faces of Δ for $i = 2, 3, 4, 5, 6, 7, 8$.

Exercise A.2.2 Let $<$ be the purely lexicographic term order. Using CoCoA, compute the generic initial ideal $\mathrm{gin}_<(M)$ and its minimal free resolution, for the ideal M in the previous Exercise.

Exercise A.2.3 Pick 100 monomials in x, y, z at random with exponents between 0 and 1000. Compute the minimal free resolution and the Hilbert series of the ideal they generate. Repeat the experiment ten times. Explain your data. Try again with more monomials...

Exercise A.2.4 Draw the *second barycentric subdivision of a triangle*. Construct a monomial ideal in $k[x, y, z]$ which has that resolution. Such ideals exist by Schnyder's Theorem.

Exercise A.2.5 Explain how the Hilbert function command in CoCoA can be used to compute the Scarf complex of a generic monomial ideal. Apply your method to compute Δ_M for

$$M = \langle a^5, b^5, c^5, d^5, ab^2c^3d^4, a^2b^3c^4d, a^3b^4cd^2, a^4bc^2d^3 \rangle.$$

The Scarf complex Δ_M is a triangulation of the tetrahedron; draw it.

Exercise A.2.6 Compute the irreducible decomposition of the monomial ideal M in the previous Exercise.

Exercise A.2.7 What is the maximum number of irreducible components of an artinian ideal generated by 10 monomials in 4 variables? Can you find an example that attains the bound?

Exercise A.2.8 Consider the non-generic monomial ideal $M = \langle x, y, z \rangle^3$. Construct at least three different free resolutions of M by *deformation of exponents*.

A.3 Exercises from Day 3

Exercise A.3.1 Find a monomial ideal I such that $(I^\vee)^\vee \neq I$. Characterize such I.

Exercise A.3.2 Alexander duality commutes with taking radicals: $\sqrt{I^\vee} = (\sqrt{I})^\vee$.

Exercise A.3.3 Let I be a monomial ideal given in terms of its minimal generators, and $\mathbf{a} \in \mathbb{N}^n$ coordinatewise bigger than the exponent vectors of the generators. What is the easiest (or fastest) method in CoCoA for calculating the Alexander dual $I^{[\mathbf{a}]}$?

Exercise A.3.4 The irrelevant ideal of the product of projective spaces $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$ is a squarefree monomial ideal M in $k[x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1]$.

(a) Find the minimal generators of M.
(b) Calculate the minimal free resolution of M in CoCoA.
(c) Interpret the Betti numbers of your resolution in (b) in terms of polytopes.
(d) Show that the minimal free resolution of M is a cellular resolution.
(e) Why is the Alexander dual of M Cohen-Macaulay?

Exercise A.3.5 Draw the minimal free resolution of the cogeneric ideal

$$\langle x^1, y^4, z^6\rangle \cap \langle x^2, y^6, z^1\rangle \cap \langle x^3, y^3, z^3\rangle \cap \langle x^4, y^5, z^2\rangle \cap \langle x^5, y^1, z^5\rangle \cap \langle x^6, y^2, z^4\rangle .$$

Check your result using CoCoA.

Exercise A.3.6 What is the maximal number of minimal generators of an intersection of 12 irreducible monomial ideals in $k[x_1, x_2, x_3, x_4]$?

Exercise A.3.7 Show that the hull resolution of $\langle x^4, y^4, x^3z, y^3z, x^2z^2, y^2z^2, xz^3, yz^3\rangle$ is minimal. What is its irreducible decomposition? Where is this information hiding in the hull complex?

Exercise A.3.8 Compute the hull resolution of the ideal

$$I = \langle x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_3, x_2x_4, x_2x_5, x_3x_4, x_3x_5, x_4x_5\rangle .$$

Can you state a general result for squarefree monomial ideals?

A.4 Exercises from Day 4

Exercise A.4.1 Let M denote the monomial module generated by all Laurent monomials $x^iy^jz^k$ with the properties that $i + j + k = 0$ and not all three coordinates of (i, j, k) are even. Draw a picture of this module. Determine the minimal free resolution of M over $k[x, y, z]$.

Exercise A.4.2 Let L be the kernel of the matrix $\begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$. Show that the hull resolution of the monomial module M_L is minimal. What happens modulo the action by the lattice L?

Exercise A.4.3 Describe the canonical module of the ring $k[t^3, t^4, t^5]$ as the quotient of a lattice module in $k[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ by a lattice action. Is there a relation to Alexander duality?

Exercise A.4.4 Using CoCoA, compute the $\mathbb{Z}$-graded Hilbert series of the algebra $k[t^{20}, t^{24}, t^{25}, t^{31}]$ in the form

$$\frac{p(t)}{(1 - t^{20})(1 - t^{24})(1 - t^{25})(1 - t^{31})} .$$

Give a polyhedral explanation for each term appearing in the polynomial $p(t)$.

Exercise A.4.5 Suppose you travel to a country whose currency has four coins valued 20, 24, 25, and 31. What is the largest amount of money which cannot be expressed by these coins?

Exercise A.4.6 Explain how the hull complex of a **generic** lattice ideal can be computed in CoCoA. Apply your procedure to compute the hull complex for the ideal I_L of the 2-dimensional sublattice L of $\mathbb{Z}^4$ spanned by the vectors $(-7, -5, 3, 8)$ and $(4, -7, 9, -1)$.

Exercise A.4.7 Compute the hull resolution for the ideal of 2×2-minors of a generic 2×4-matrix.

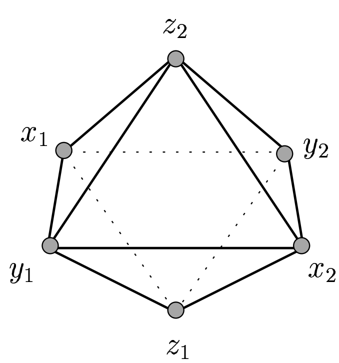
B Appendix: Solutions

Many of the solutions below were contributed by students of the 1999 COCOA Summer School in Turin, Italy, although their solutions have been edited for clarity and for technical reasons. When the printed solution has more than one author, the contributing author is labeled with an asterisk (*). When more than one group contributed solutions which were subsequently amalgamated by the authors of these lectures, each group has been parenthesized. All exercises which include computations refer to the program CoCoA [8].

B.1 Solutions for Day 1

Solution B.1.1 *By (Bahman Engheta, Leah Gold*, Ed Mosteig), (Kimberly Presser*), (Gerhard Quarg*), and (Carolyn Yackel*)*

In fact, these ideals appear in Lecture VI (Example 6.3). Here, we prefer to label the vertices of the octahedron as indicated in the picture:



In order to get the ideal I_Δ, write down the minimal nonfaces of the octahedron, which are the edges connecting variables on the same axis:

$$I_\Delta = \langle x_1x_2, y_1y_2, z_1z_2 \rangle .$$

For $I_{\Delta^\vee}$, just switch the roles of the minimal generators and prime components:

$$\begin{aligned} I_{\Delta^\vee} &= \langle x_1, y_1\rangle \cap \langle y_1, y_2\rangle \cap \langle z_1, z_2\rangle \\ &= \langle x_2y_2z_2, x_1y_2z_2, x_2y_1z_2, x_1y_1z_2, x_2y_2z_1, x_1y_2z_1, x_2y_1z_1, x_1y_1z_1\rangle. \end{aligned}$$

The Hilbert series for S/I_Δ can be calculated by hand from the polytopes as described in Lecture I or read off of the minimal free resolution calculated below:

$$\frac{1 - x_1x_2 - y_1y_2 - z_1z_2 + x_1x_2y_1y_2 + x_1x_2z_1z_2 + y_1y_2z_1z_2 - x_1x_2y_1y_2z_1z_2}{(1-x_1)(1-x_2)(1-y_1)(1-y_2)(1-z_1)(1-z_2)} .$$

We get the coarse Hilbert series by setting all variables equal to t:

$$\frac{1 - 3t^2 + 3t^4 - t^6}{(1-t)^6} = \frac{1 + 3t + 3t^2 + t^3}{(1-t)^3}.$$

Similarly for $I_{\Delta^\vee}$: the numerator of the Hilbert series of $S/I_{\Delta^\vee}$ is

$$\begin{aligned}1 \;&-\; x_1y_1z_1 - x_2y_1z_1 - x_1y_2z_1 - x_2y_2z_1 - x_1y_1z_2 - x_2y_1z_2 - x_1y_2z_2 - x_2y_2z_2\\ &+\; x_2y_2z_1z_2 + x_1y_2z_1z_2 + x_2y_1z_1z_2 + x_1y_1z_1z_2 + x_2y_1y_2z_2 + x_1y_1y_2z_2\\ &+\; x_1x_2y_2z_2 + x_1x_2y_1z_2 + x_2y_1y_2z_1 + x_1y_1y_2z_1 + x_1x_2y_2z_1 + x_1x_2y_1z_1\\ &-\; x_2y_1y_2z_1z_2 - x_1y_1y_2z_1z_2 - x_1x_2y_2z_1z_2 - x_1x_2y_1z_1z_2 - x_1x_2y_1y_2z_2\\ &-\; x_1x_2y_1y_2z_1 + x_1x_2y_1y_2z_1z_2\,,\end{aligned}$$

while the coarse Hilbert series of $I_{\Delta^\vee}$ is

$$\frac{1-8t^3+12t^4-6t^5+t^6}{(1-t)^6} = \frac{1+2t+3t^2-4t^3+t^4}{(1-t)^4}.$$

One may calculate the coarse Hilbert series with CoCoA as follows:

```
Use S ::= Q[x[1..2]y[1..2]z[1..2]];
I := Ideal(x[1]x[2],y[1]y[2],z[1]z[2]);
IDual := Intersection(Ideal(x[1],x[2]),Ideal(y[1],y[2]),
                      Ideal(z[1],z[2]));
IDual;
Ideal(x[2]y[2]z[2], x[1]y[2]z[2], x[2]y[1]z[2], x[1]y[1]z[2],
x[2]y[2]z[1], x[1]y[2]z[1], x[2]y[1]z[1], x[1]y[1]z[1])
-------------------------------
Poincare(S/I);
(1 + 3x[1] + 3x[1]^2 + x[1]^3) / (1-x[1])^3
-------------------------------
Poincare(S/IDual);
(1 + 2x[1] + 3x[1]^2 - 4x[1]^3 + x[1]^4) / (1-x[1])^4
-------------------------------
```

One may use CoCoA to calculate the minimal free resolutions. Continuing from above:

```
Res(S/I);
0 --> S(-6) --> S^3(-4) --> S^3(-2) --> S
-------------------------------
Describe It;

Mat[
  [z[1]z[2], y[1]y[2], x[1]x[2]]
]
Mat[
  [y[1]y[2], x[1]x[2], 0],
  [-z[1]z[2], 0, x[1]x[2]],
  [0, -z[1]z[2], -y[1]y[2]]
]
```

```
Mat[
  [x[1]x[2]],
  [-y[1]y[2]],
  [z[1]z[2]]
]
-------------------------------
```

Using monomial matrices the minimal free resolution of S/I_Δ is

$$0 \longleftarrow S \xleftarrow{\;\begin{array}{c|ccc} & z_1z_2 & y_1y_2 & x_1x_2 \\ \hline 1 & 1 & 1 & 1 \end{array}\;} S^3 \xleftarrow{\;\begin{array}{c|ccc} & y_1y_2z_1z_2 & x_1x_2z_1z_2 & x_1x_2y_1y_2 \\ \hline z_1z_2 & 1 & 1 & 0 \\ y_1y_2 & -1 & 0 & 1 \\ x_1x_2 & 0 & -1 & -1 \end{array}\;} S^3 \xleftarrow{\;\begin{array}{c|c} & x_1x_2y_1y_2z_1z_2 \\ \hline y_1y_2z_1z_2 & 1 \\ x_1x_2z_1z_2 & -1 \\ x_1x_2y_1y_2 & 1 \end{array}\;} S^1 \longleftarrow 0$$

The alternating sum of the column labels is the numerator of the Hilbert series of S/I_Δ.

The minimal free resolution of $S/I_{\Delta^\vee}$ is

$$0 \longleftarrow S \longleftarrow S^8 \longleftarrow S^{12} \longleftarrow S^6 \longleftarrow S^1 \longleftarrow 0\,.$$

The matrices are too big to fit comfortably on the page (either in monomial matrix form or usual form). However, we note that the monomial matrices are filled with the boundary maps of the cube (Proposition 5.4)—observe the reflection of this fact in the total Betti numbers $(1, 8, 12, 6, 1)$ of $S/I_{\Delta^\vee}$.

Use Hochster's formula (Theorem 1.8), $\beta_{i,\sigma}(S/I_\Delta) = \dim_k \widetilde{H}^{|\sigma|-i-1}(\Delta|_\sigma; k)$, to relate the Betti numbers with simplicial homology. For each incidence vector σ—which we identify with a subset of the vertices of the octahedron—the Betti number $\beta_{i,\sigma}(S/I_\Delta)$ is 1 precisely when $\mathbf{x}^\sigma$ is a column label for the i-th monomial matrix appearing in the minimal free resolution of S/I_Δ; otherwise it is 0. Thus, for instance, from the column label $x_1x_2z_1z_2$ appearing in the middle matrix of the resolution displayed earlier we see that

$$\beta_{2,\sigma} = \dim_k \widetilde{H}^{|\sigma|-3}(\Delta|_\sigma; k) = \dim_k \widetilde{H}^1(\Delta|_\sigma; k) = \dim_k \widetilde{H}_1 \Delta|_\sigma; k) = 1.$$

Here, σ corresponds to the four vertices of the octahedron labeled x_1, x_2, z_1, and z_2. Hence, $\Delta|_\sigma$ is the boundary of a square. Since the column label $x_1x_2z_1z_2$ only appears once in the resolution, Hochster's formula shows that the reduced homology of $\Delta|_\sigma$ is 0 in every other homological degree, as expected.

When $\sigma = \mathbf{1}$ is defined by $\mathbf{x}^\sigma = x_1x_2y_1y_2z_1z_2$, we get $\Delta|_\sigma = \Delta$. Using Hochster's formula, we see that $\widetilde{H}_2(\Delta; k) \cong k$ and the other homology is zero.

Solution B.1.2

Although CoCoA doesn't readily calculate $\mathbb{Z}^n$-graded Betti numbers, the only Betti numbers we need to find the homology of a simplicial complex Δ are $\beta_{i,(1,\ldots,1)}(S/I_\Delta)$, and these are the only Betti numbers in total ($\mathbb{Z}$-graded) degree n = #variables.

Therefore, we can calculate $\dim_k \widetilde{H}_i(\Delta; k) = \beta_{n-i-1,(1,\ldots,1)}(S/I_\Delta)$ (Theorem 1.8) simply by reading off the number of summands of the form $S(-n)$ at the i^{th} step in the minimal free resolution of S/I_Δ.

Solution B.1.3

First create the 5×5 chessboard complex and calculate its Hilbert series:

```
N := 5;
Use S ::= Q[x[1..N,1..N]];

G := []; -- generators for the ideal
For I := 1 To N Do
  For J := 1 To N Do
    For L := 1 To N Do
      If J <> L Then
G := Concat(G,[x[I,J]x[I,L],x[J,I]x[L,I]]);
      End;
    End;
  End;
End;
Len(Set(G)); -- just to make sure we get 100 monomials
100
-------------------------------
IDelta := Ideal(G); -- create the ideal
Poincare(S/IDelta);
(1 + 20x[1,1] + 110x[1,1]^2 + 140x[1,1]^3 -
95x[1,1]^4 - 56x[1,1]^5) / (1-x[1,1])^5
-------------------------------
```

Now compute the f-vector:

```
-- function for computing the f-vector
Define FVector(I)
  L := HVector(CurrentRing()/I);
  D := Dim(CurrentRing()/I);
  Using Qt Do
    H := Sum([L[N]t^(N-1) | N In 1..Len(L)]);
    E := Subst(H,t,1);
    F := [E];
    For N := 0 To D-1 Do
      H := (H-E*t^(D-N))/(1-t);
      E := Subst(H,t,1);
      F := Concat([E],F);
    End;
  End;
```

```
    Return [LC(X) | X In F];  -- LC used to suppress printing of 'Qt'
End;

FVector(IDelta);
[1, 25, 200, 600, 600, 120]
-------------------------------
```

Hence, there are 25 vertices, 200 edges, 600 two-dimensional faces, etc.

In theory we should be able to calculate the homology using the CoCoA command `Res(S/IDelta)` and looking for the number of summands of the form $S(-25)$ in each degree. Unfortunately, the calculation does not terminate successfully in CoCoA. For the 3×3 case, go back and set `N := 3`. We get

```
FVector(IDelta);
[1, 9, 18, 6]
-------------------------------
Res(S/IDelta);
0 --> S^4(-9) --> S^27(-8) --> S^72(-7) --> S^9(-5)(+)S^90(-6) -->
S^45(-4)(+)S^45(-5) --> S^48(-3)(+)S^9(-4) --> S^18(-2) --> S
-------------------------------
```

which according to Solution B.1.2 means that $\dim_k \widetilde{H}_1(\Delta; k) = \beta_7(S/I_\Delta) = 4$ and the other homology groups are zero.

Solution B.1.4

Here is some CoCoA code to accomplish the task. Unfortunately, this code will probably not finish in a reasonable amount of time, except for the reverse lex gin.

```
S ::= Q[x,y,z];
T ::= Q[x[1..10]];  --Default order Reverse Lex
U ::= Q[x[1..10]], Lex;
Use T;

Using S Do                            --Create a matrix whose columns
  M := Ideal(Indets())^3;             --are the exponent vectors of the
  L := [Log(J) | J In MinGens(M)];    --10 degree 3 monomials
  L := Transposed(L);
End;

I := Toric(L);  --Form the toric ideal for this matrix

Define Gin(I)
  RandLin(N) := [Sum([Rand(-100,100) J | J In Indets()]) | M In 1..N];
  J := Eval(I,RandLin(NumIndets()));
  Return(LT(J));
End;
```

The following is the result of taking the reverse lex initial ideal:

```
Use T;
Gin(I);
Ideal(x[1]^2, x[1]x[2], x[2]^2, x[1]x[3], x[2]x[3], x[3]^2, x[1]x[4],
x[2]x[4], x[3]x[4], x[4]^2, x[1]x[5], x[2]x[5], x[3]x[5], x[4]x[5],
x[5]^2, x[1]x[6], x[2]x[6], x[3]x[6], x[4]x[6], x[5]x[6], x[6]^2,
x[1]x[7], x[2]x[7], x[3]x[7], x[4]x[7], x[5]x[7], x[6]x[7], x[7]^3)
```

Solution B.1.5 *By Carolyn Yackel*

Any nonradical ideal I whose radical is $\sqrt{I} = \mathfrak{m} = \langle x_1, \ldots, x_n \rangle$ cannot be an initial ideal of a homogeneous prime. Indeed, any homogeneous ideal J having initial ideal I is artinian, and therefore has support on $\mathfrak{m}$. The Hilbert function of such a J gives away the fact that J is not radical. In particular, I can be Borel-fixed—for instance, I can be a power of $\mathfrak{m}$. In general, Borel-fixed ideals which are primary to ideals generated by initial subsets of the variables will have the same problems.

Solution B.1.6 *By Mircea Mustaţă and Greg Smith**

Every initial ideal can is the result of taking an initial with respect to a *weight order* [12, Chapter 15]. We start by finding a list of weight vectors which give different initial ideals. Without loss of generality, it is enough to consider vectors of the form $(A, 2, 1)$: because the ideal is homogeneous the initial ideal remains the same if one adds or subtracts $(1, 1, 1)$ and if one multiplies by a scalar. We may assume the first entry of $(A, 2, 1)$ is larger than the second and that the second is larger than the third, as long as we remember to apply all permutations of the variables when we're done.

```
LL := [];
For A := 3 To 20 Do
    W := Mat[[A,2,1]];
    S ::= Q[abc], Weights(W);
    Using S Do
        L := GenericPoints(11);
        L := Tail(Tail(Tail(Tail(L))));
        I := IdealOfProjectivePoints(L);
        J := [Log(X) | X In Gens(LT(I))];
        If Not (J IsIn LL) Then
            Append(LL, J);
            Print([A,2,1]);
        End;
     End;
End;
Print(LL);
```

From this we obtain four vectors: $(3, 2, 1)$, $(4, 2, 1)$, $(6, 2, 1)$ and $(8, 2, 1)$. Now we determine the interval associated to each vector, using the fact that if two different

A return the same initial ideal then so do all intermediate A. For instance, let's see what happens when we increase A from 3 to 5.

```
W := [3,2,1];
S ::= Z/(32003)[abc], Weights(W);
Define TestWeight()
  Using S Do
    L := GenericPoints(11);
    L := Tail(Tail(Tail(Tail(L))));
    I := IdealOfProjectivePoints(L);
    G := ReducedGBasis(S/I);
  End;
  Return(LT(G));
End;

TestWeight();
[S :: ab^2, S :: a^2b, S :: a^3, S :: b^4]
-------------------------------
W := [301,200,100];
S ::= Z/(32003)[abc], Weights(W);
TestWeight();
[S :: a^2c, S :: a^2b, S :: a^3, S :: ab^2c, S :: ab^3, S :: b^5]
-------------------------------
W := [5,2,1];
S ::= Z/(32003)[abc], Weights(W);
TestWeight();
[S :: a^2c, S :: a^2b, S :: a^3, S :: ab^2c, S :: ab^3, S :: b^5]
-------------------------------
W := [501,200,100];
S ::= Z/(32003)[abc], Weights(W);
TestWeight();
[S :: a^2c, S :: a^2b, S :: a^3, S :: ab^2c, S :: ab^3, S :: abc^3,
 S :: b^6]
-------------------------------
```

Note that we had to use `[301,200,100]` in place of `[3.01,2,1]` since CoCoA requires integers for weights. The above CoCoA session shows that the initial ideal is constant for $3 < \mathtt{A} \leq 5$, but changes when $\mathtt{A} = 3$ or $\mathtt{A} > 5$. Doing this for the four critical values of A found above, the intervals are

$$2 \leq \mathtt{A} \leq 3, \quad 3 < \mathtt{A} \leq 5, \quad 5 < \mathtt{A} \leq 7, \quad 7 < \mathtt{A}\,.$$

One checks easily that applying the permutations of the variables to the initial ideals for $\mathtt{A} = 3, 5, 7, 8$ doesn't produce any ideal twice (look at what happens to the powers of variables in these ideas). Therefore, there are $6 \cdot 4 = 24$ distinct initial ideals.

Solution B.1.7

Distractions are covered to some extent in the other lectures in this volume, so we'll just give some sketches and references. A finite generating set is obtained from the minimal generators of M by replacing each occurrence of $x_i^{b_i}$ by $x_i(x_i - 1)(x_i - 2)\cdots(x_i - b_i + 1)$. Thus D_M is generated by the *distractions* $D(m)$ of the monomials m generating M, where, for example,

$$D(x^3y^4z) = x(x-1)(x-2)y(y-1)(y-2)(y-3)z\,.$$

Every scalar in k is less than every variable in a term order, so the initial term of a distraction $D(m)$ is m itself. This implies that M is contained in any initial ideal of D_M. If M is artinian then M is the initial ideal of D_M with respect to all term orders because S/M and S/D_M have the same length (k-vector space dimension). In particular, the distraction of the minimal generating set of M is a Gröbner basis if M is artinian. For general M, introduce powers of the variables so high that they cannot interfere with Buchberger's algorithm on the distractions of the minimal generators of M. Since we know for artinian M that the distractions form a Gröbner basis, it must be that Buchberger's criterion applies without the presence of the distractions of the introduced high powers of variables. Therefore, $\{D(m) \mid m \text{ is a minimal generator of } M\}$ is a reduced Gröbner basis for every term order.

The prime decomposition of $D(M)$ is given by the set of *standard pairs* [35], which refine the information in irreducible decomposition. However, our CoCoA routine to calculate the arithmetic degree will be based simply on irreducible decomposition and some counting, without listing the standard pairs.

The ideal `M` used as the test case below is the Permutohedron ideal of Example 6.12, whose irreducible components have varying dimensions. It is also a good idea to test its Alexander dual,

```
Ideal(xyz,x^2y^2,y^2z^2,x^2z^2,x^3,y^3,z^3)},
```

which has only one primary component that splits into lots of irreducible components.

```
M := Ideal(xy^2z^3,x^2y^3z,x^3yz^2,x^2yz^3,x^3y^2z,xy^3z^2);
A:= Log(LCM(Gens(M)));
```

The next function `Pos` is a CoCoA command for supp(`L`), when applied to vectors `L` $\in \mathbb{N}^n$. The command `Artinian` throws in the high coordinate of `A` to the exponent vector `B` whenever `B` has a zero there. We'll need this to compute the irreducible decomposition via Alexander duality, and to reduce everything to finite-dimensional k-vector spaces, later on.

```
Pos(L):= [Min(Concat(X,[1])) | X In List(Transposed(Mat(L)))];

Define Artinian(A,B)
  C := [];
  For N := 1 To Len(A) Do
```

```
    If B[N] = 0 Then Output := A[N]+1
      Else Output := B[N];
    End;
    Append(C,Output);
  End;
  Return(C);
End;
```

The following `AlexDual` command is one of those submitted by Robert Forkel for Solution B.3.3.

```
Define AlexDual(...)
  N:=NumIndets();
  I:=Minimalized(ARGV[1]);
  If Len(ARGV)=2 Then A:=ARGV[2]
  Else A:=Log(LCM(Gens(I)))
  End;
  I:=Mat([[1|K In 1..N]-Log(M)+A|M In Gens(ARGV[1])]);
  If Len(ARGV) > 2 Then Error('Too many arguments in AlexDual') End;
  L:=[];
  For K:=1 To Len(I) Do
  Append(L,[(Indet(M))^I[K,M]|M In 1..N And I[K,M]<=A[M]]);
  End;
  Return IntersectionList([Ideal(M)|M In L])
End;
```

Getting the irreducible decomposition from the Alexander dual is easy.

```
BToTheA(A,B) := A + [1 | X In Indets()] - Artinian(A,B);
Define IrrDecomp(I)
  J := AlexDual(I);
  A := Log(LCM(Gens(J)));
  Return([BToTheA(A,Log(B)) | B In Gens(J)]);
End;

Irr := IrrDecomp(M); Irr;
[[0, 0, 0, 1], [0, 3, 3, 3], [0, 0, 1, 0], [0, 1, 0, 0],
               [0, 0, 2, 2], [0, 2, 0, 2], [0, 2, 2, 0]]
-------------------------------
```

We have now fixed an ordering of the irreducible components. For each Q primary to P on the list we write down (i) the other components on the list whose associated primes are strictly contained in P, plus (ii) those associated to P which appear after Q.

```
Divides(Y,X):= Mod(LogToTerm(Y),LogToTerm(X)) = 0;
                                                --True if X divides Y
```

```
PartOrder(Irr):=
[[Irr[N] | N In 1..Len(Irr) And Divides(Pos([Irr[M]]),Pos([Irr[N]]))
  And (Pos([Irr[M]])<>Pos([Irr[N]]) Or N >= M)] | M In 1..Len(Irr)];

PO := PartOrder(Irr); PO;
[[[0, 0, 0, 1]],
[[0, 0, 0, 1], [0, 3, 3, 3], [0, 0, 1, 0], [0, 1, 0, 0],
               [0, 0, 2, 2], [0, 2, 0, 2], [0, 2, 2, 0]],
[[0, 0, 1, 0]],
[[0, 1, 0, 0]],
[[0, 0, 0, 1], [0, 0, 1, 0], [0, 0, 2, 2]],
[[0, 0, 0, 1], [0, 1, 0, 0], [0, 2, 0, 2]],
[[0, 0, 1, 0], [0, 1, 0, 0], [0, 2, 2, 0]]]
-------------------------------
```

For each irreducible component in the list corresponding to Q, we add artinian generators coming from `A`, except that we ignore the variables not in P.

```
A(A,Q):= Artinian(A,[1|X In Indets()]-Pos([Q]))-[1 | X In Indets()];

Art(A,Ir,PO):=[[Artinian(A(A,Ir[N]),Q)|Q In PO[N]]|N In 1..Len(PO)];
APO := Art(A,Irr,PO);
```

The intersection of these "artinianized" components has finite codimension as a k-vector subspace of the polynomial ring. Omitting the artinianized version of Q itself from the intersection still leaves us with an ideal of finite codimension, less than the codimension with Q included. The difference of these codimensions is the contribution from the irreducible component Q.

```
M(B):= Ideal([Indet(N)^B[N] | N In 1..NumIndets()]);

WithQ(APO):=[IntersectionList([M(B)|B In X]) | X In APO]; WithQ(APO);
[Ideal(t, x, y, z), Ideal(t, z^4, y^4, x^4, xy^2z^3, x^2yz^3, xy^3z^2,
 x^3yz^2, x^2y^3z, x^3y^2z), Ideal(t, x, y, z), Ideal(t, x, y, z),
 Ideal(x, t, yz^2, y^2z, z^4, y^4), Ideal(y, t, xz^2, x^2z, z^4, x^4),
 Ideal(z, t, xy^2, x^2y, y^4, x^4)]
-------------------------------

WithoutQ(A,APO,Irr):=[IntersectionList([M(B) | B In APO[N] And
                      B <> Artinian(A(A,Irr[N]),Irr[N])])
                                          | N In 1..Len(APO)];
WithoutQ:= WithoutQ(A,APO,Irr); WithoutQ;
[[ ], Ideal(t, z^4, y^4, x^4, xy^2z^2, x^2yz^2, x^2y^2z), [ ], [ ],
 Ideal(x, t, yz, z^4, y^4), Ideal(z^4, x^4, y, t, xz),
 Ideal(xy, y^4, x^4, z, t)]
-------------------------------
```

```
Define Mult(I)
  If Type(I) = IDEAL Then Mult := Multiplicity(CurrentRing()/I)
    Else Mult := 0;
  End;
  Return(Mult);
End;

DegWithQ:= [Mult(K) | K In WithQ(APO)]; DegWithQ;
[1, 48, 1, 1, 8, 8, 8]
-------------------------------
DegWithoutQ:= [Mult(K) | K In WithoutQ]; DegWithoutQ;
[0, 44, 0, 0, 7, 7, 7]
-------------------------------
Degrees:= DegWithQ - DegWithoutQ; Degrees;
[1, 4, 1, 1, 1, 1, 1]
-------------------------------
ArithDeg:= Sum(Degrees); ArithDeg;
10
-------------------------------
```

The contribution of a single irreducible component to the arithmetic degree is only well-defined after choosing an ordering of the irreducible components. However, the total contribution from a given *associated* prime is well defined.

B.2 Solutions for Day 2

Solution B.2.1 *By (Bahman Engheta, Leah Gold*, Ed Mosteig), (Kimberly Presser*), (Gerhard Quarg*), and (Carolyn Yackel*)*

The following CoCoA code uses the function `FVector`, defined as in Solution B.1.3. We assume this function has already been entered into CoCoA.

```
Use S ::= Q[x[1..4]y[1..4]z[1..4]];
-- the polarization appears next:
IDelta := Ideal(x[1]x[2]x[3]x[4],y[1]y[2]y[3]y[4],z[1]z[2]z[3]z[4],
x[1]x[2]x[3]y[1]y[2]z[1],x[1]y[1]y[2]y[3]z[1]z[2],
x[1]x[2]y[1]z[1]z[2]z[3]);
FVector(IDelta);
[1, 12, 66, 220, 492, 768, 837, 609, 264, 51]
-------------------------------
```

Thus, Δ has 12 vertices, 66 edges, etc.

Gerhard Quarg found another way to calculate the f-vector. First calculate the Hilbert series.

```
Poincare(S/IDelta);
(1 + 3x[1] + 6x[1]^2 + 10x[1]^3 + 12x[1]^4 + 12x[1]^5 + 7x[1]^6) /
(1-x[1])^9
-------------------------------
```

Now use the formula

$$H(t) = \frac{1}{(1-t)^{12}} \sum_{i=0}^{12} f_{i-1}\, t^i (1-t)^{12-i}.$$

Multiply by $(1-t)^{12}$ to get the numerator we need and compute the f vector using the CoCoA function `GenRepr`. The indeterminate `x[1]` plays the role of t.

```
F := (1-x[1])^12(1 + 3x[1] + 6x[1]^2 + 10x[1]^3 + 12x[1]^4 +
12x[1]^5 + 7x[1]^6) / (1-x[1])^9;
GenRepr(F,Ideal([x[1]^I (1-x[1])^(12-I) | I In 0..12]));
[1, 12, 66, 220, 492, 768, 837, 609, 264, 51, 0, 0, 0]
-------------------------------
```

Solution B.2.2 *By Kimberly Presser*

```
Define Generic(I)
Help 'The CurrentRing() should be Q[x[1..N]] where N is the number of
      indeterminates.';
J:=I;
N:=NumIndets();
   For K:=1 To N Do
     F:=DensePoly(1);
     L:=Randomized(F);
     J:=Subst(J,x[K],L);
   End;
   Gin:=LT(J);
   Return Gin;
End;

Use R::=Q[x[1..3]],Lex;
M:=Ideal(x[1]^4,x[2]^4,x[3]^4,x[1]^3x[2]^2x[3],x[1]x[2]^3x[3]^2,
         x[1]^2x[2]x[3]^3);
G:=Generic(M);
G;
Ideal(x[1]^4, x[1]^3x[2], x[1]^3x[3], x[1]^2x[2]^3, x[1]^2x[2]^2x[3],
x[1]^2x[2]x[3]^2, x[1]^2x[3]^4, x[1]x[2]^5, x[1]x[2]^4x[3],
x[1]x[2]^3x[3]^2, x[1]x[2]^2x[3]^3, x[1]x[2]x[3]^4, x[1]x[3]^5,
x[2]^7, x[2]^6x[3], x[2]^5x[3]^2, x[2]^4x[3]^3, x[2]^3x[3]^4,
x[2]^2x[3]^5, x[2]x[3]^6, x[3]^7)
```

```
------------------------------
Res(R/G);
0 --> R(-6)(+)R^2(-7)(+)R^6(-8)(+)R^7(-9) -->
R^3(-5)(+)R^5(-6)(+)R^13(-7)(+)R^15(-8) -->
R^3(-4)(+)R^3(-5)(+)R^7(-6)(+)R^8(-7) --> R
------------------------------
```

Solution B.2.3

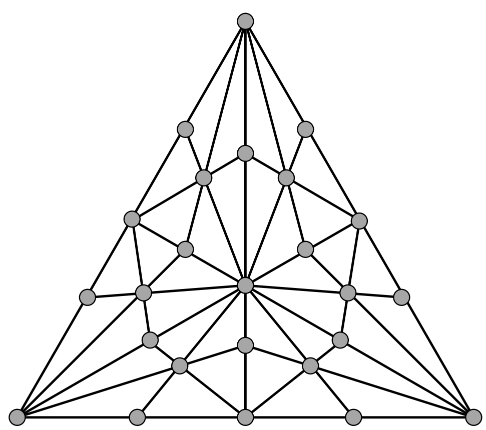
Clearly we will not be printing the solution to this one. However, the reader may find the following command useful.

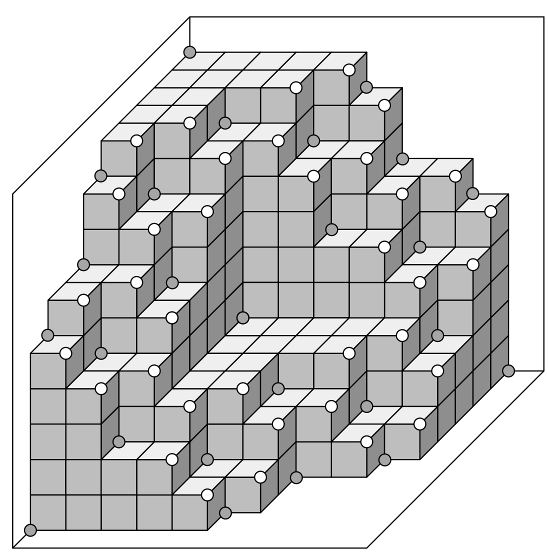
```
RandMon(N,Exp) := [LogToTerm([Rand(0,Exp) | L In 1..NumIndets()])
                    | M In 1..N];
```

It generates `N` random monomials with exponents between `0` and `Exp`.

Solution B.2.4 *By Burckhard Zimmerman*

The second barycentric subdivision of a triangle is at left below, and resolves the generic ideal with the staircase at right. Recall that the tree ideal (Section 5.3) has cellular minimal resolution supported on the *first* barycentric subdivision of a simplex. The above staircase is constructed roughly by arranging a bunch of tree ideals around a central inner corner. This arrangement is easiest to see by looking at the six hexagonal groups of white dots = irreducible components. [The first author of these Lectures was quite surprised when Mr. Zimmerman found precisely the same solution to this problem as he did. Most people think that the staircase is aesthetically more pleasing upside-down, representing the cogeneric Alexander dual.]





Solution B.2.5

CoCoA's standard use of `Poincare` is to give the $\mathbb{Z}$-graded Hilbert-Poincare series of a quotient S/M in reduced terms, so that the denominator is $(1-\text{variable})^d$, where $d =$

$\dim(S/M)$. If we could somehow convince CoCoA to give us the multigraded Hilbert series, the Scarf complex would consist of the monomials in the numerator. It turns out that CoCoA does have a feature for using `Poincare` when the variables have linearly independent *weights*, and this allows CoCoA to use finer gradings. Unfortunately, the first row of the `WeightsMatrix` must consist of positive integers, so we are not allowed to assign the weight $\mathbf{e}_i \in \mathbb{Z}^n$ to the i^{th} variable. As a meager substitute, we add a new homogenizing variable at the top of the list, and proceed to forget about it later. In our case, we need 4 variables a, b, c, and d, so we use t as the "zeroth" variable:

```
Use S ::= Q[t,a,b,c,d], Weights(Mat([[1,1,1,1,1],
                                     [0,1,0,0,0],
                                     [0,0,1,0,0],
                                     [0,0,0,1,0],
                                     [0,0,0,0,1]]));
M:=Ideal(a^5,b^5,c^5,d^5,ab^2c^3d^4,a^2b^3c^4d,a^3b^4cd^2,a^4bc^2d^3);
P:=Poincare(S/M);
```

It looks like `P` is a rational function, but really it's a list of coefficients and exponent vectors, as the command

```
Describe(P);
```

demonstrates. The elements of Δ_M are uniquely determined by their vector labels, which are obtained from the exponent vectors in the numerator of the Hilbert series by leaving off the exponent of `t`:

```
N := [Tail(X[2]) | X In P[1]];
N;
[[5, 5, 5, 1], [5, 5, 4, 2], [5, 4, 4, 3], [5, 3, 5, 3], [4, 4, 4, 4],
 [3, 5, 4, 4], [4, 3, 5, 4], [2, 5, 5, 4], [5, 5, 1, 5], [5, 4, 2, 5],
 [4, 4, 3, 5], [3, 5, 3, 5], [5, 1, 5, 5], [4, 2, 5, 5], [1, 5, 5, 5],
 [5, 5, 5, 0], [5, 5, 4, 1], [5, 4, 4, 2], [5, 3, 4, 3], [4, 4, 4, 3],
 [4, 3, 5, 3], [4, 4, 3, 4], [3, 5, 3, 4], [4, 3, 4, 4], [3, 4, 4, 4],
 [2, 5, 4, 4], [4, 2, 5, 4], [1, 5, 5, 4], [5, 5, 0, 5], [5, 4, 1, 5],
 [4, 4, 2, 5], [3, 4, 3, 5], [5, 0, 5, 5], [4, 1, 5, 5], [0, 5, 5, 5],
 [5, 3, 5, 1], [3, 5, 4, 2], [5, 4, 2, 3], [4, 3, 4, 3], [5, 1, 5, 3],
 [3, 4, 3, 4], [2, 3, 5, 4], [3, 5, 1, 5], [4, 2, 3, 5], [1, 5, 3, 5],
 [5, 3, 4, 1], [2, 5, 5, 1], [5, 5, 1, 2], [3, 4, 4, 2], [4, 4, 2, 3],
 [4, 1, 5, 3], [4, 2, 3, 4], [1, 5, 3, 4], [2, 3, 4, 4], [3, 4, 1, 5],
 [5, 1, 2, 5], [1, 2, 5, 5], [2, 5, 4, 1], [5, 4, 1, 2], [1, 2, 5, 4],
 [4, 1, 2, 5], [2, 3, 5, 1], [3, 5, 1, 2], [5, 1, 2, 3], [1, 2, 3, 5],
 [5, 5, 0, 0], [5, 0, 5, 0], [0, 5, 5, 0], [2, 3, 4, 1], [3, 4, 1, 2],
 [4, 1, 2, 3], [1, 2, 3, 4], [5, 0, 0, 5], [0, 5, 0, 5], [0, 0, 5, 5],
 [5, 0, 0, 0], [0, 5, 0, 0], [0, 0, 5, 0], [0, 0, 0, 5], [0, 0, 0, 0]]
```

If you want to know how the faces of Δ_M are described as least common multiples of minimal generators, just determine, for each element `N[I]` in the list `N`, which generators of `M` divide the monomial $(a,b,c,d)^{\texttt{N[I]}}$:

```
G := Gens(M);
D := [[J In 1..Len(G) | Mod(LogToTerm(Concat([0],N[I])),G[J]) = 0]
       | I In 1..Len(N)];
D;
[[1, 2, 3, 6], [1, 2, 6, 7], [1, 6, 7, 8], [1, 3, 6, 8], [5, 6, 7, 8],
 [2, 5, 6, 7], [3, 5, 6, 8], [2, 3, 5, 6], [1, 2, 4, 7], [1, 4, 7, 8],
 [4, 5, 7, 8], [2, 4, 5, 7], [1, 3, 4, 8], [3, 4, 5, 8], [2, 3, 4, 5],
 [1, 2, 3], [1, 2, 6], [1, 6, 7], [1, 6, 8], [6, 7, 8], [3, 6, 8],
 [5, 7, 8], [2, 5, 7], [5, 6, 8], [5, 6, 7], [2, 5, 6], [3, 5, 8],
 [2, 3, 5], [1, 2, 4], [1, 4, 7], [4, 7, 8], [4, 5, 7], [1, 3, 4],
 [3, 4, 8], [2, 3, 4], [1, 3, 6], [2, 6, 7], [1, 7, 8], [1, 3, 8],
 [3, 5, 6], [2, 4, 7], [4, 5, 8], [2, 4, 5], [2, 3, 6], [1, 2, 7],
 [1, 4, 8], [3, 4, 5],   [6, 8], [5, 7], [1, 6], [6, 7], [7, 8],
 [3, 8], [5, 8], [2, 5], [5, 6], [4, 7], [2, 6], [1, 7], [3, 5],
 [4, 8], [3, 6], [2, 7], [1, 8], [4, 5], [1, 2], [1, 3], [2, 3],
 [1, 4], [2, 4], [3, 4], [6], [7], [8], [5], [1], [2], [3], [4], [ ]]
```

Don't forget about the zeroth variable `t`, which is the reason for the `Concat`! This Scarf complex is the Schlegel diagram for the 4-dimensional cross-polytope (higher-dimensional analog of the octahedron), which can be expressed abstractly as the convex hull of the points $\pm\mathbf{e}_i$ for $i \in \{1, 2, 3, 4\}$.

Solution B.2.6 *By Carolyn Yackel*

All we need is the exponent vectors on the facets of the Scarf complex from the previous exercise. Continuing the CoCoA session from there, we want the elements of `N` whose corresponding element of `D` has length 4:

```
Irr := [N[I] | I In 1..Len(N) And Len(D[I]) = 4];
Irr;
[[5, 5, 5, 1], [5, 5, 4, 2], [5, 4, 4, 3], [5, 3, 5, 3], [4, 4, 4, 4],
 [3, 5, 4, 4], [4, 3, 5, 4], [2, 5, 5, 4], [5, 5, 1, 5], [5, 4, 2, 5],
 [4, 4, 3, 5], [3, 5, 3, 5], [5, 1, 5, 5], [4, 2, 5, 5], [1, 5, 5, 5]]
```

These $15 = 16 - 1 = 2^4 - 1$ facets of the cross-polytope correspond to all but one of the vertices of the hypercube, which is polar to the cross-polytope.

Solution B.2.7 *By Carolyn Yackel*

We may assume the ideal is generic, since deformation only makes the Betti numbers go up, and the number of irreducible components of an artinian ideal is the last nonzero Betti number. An upper bound for the number of irreducible components of a monomial ideal in n variables with r generators is given by one less than the number of facets of the cyclic n-polytope with r vertices. By a theorem of G. Agnarsson [1], this upper bound is attained if $n \leq 3$ or $n = 4$ and $r \leq 12$. Thus the upper bound 34 for $n = 4$ and $r = 10$ is attained by some monomial ideal. For example, leave out the last two generators of the ideal in Example 4.15.

Solution B.2.8

This is best done by the reader, following the model of Example 4.12.

B.3 Solutions for Day 3

Solution B.3.1 *By (Abdul Jarrah*), (Bahman Engheta, Leah Gold*, and Ed Mosteig)*

Given a monomial ideal I, let $\mathbf{a}_I$ be the exponent on the least common multiple of the minimal generators of I, so $I^\vee = I^{[\mathbf{a}_I]}$. It follows directly from the definition of Alexander dual ideal that $\mathbf{a}_{I^\vee} \preceq \mathbf{a}_I$. The definition also implies that if $\mathbf{a} \neq \mathbf{a}'$ are both $\succeq \mathbf{a}_I$, then $I^{[\mathbf{a}]} \neq I^{[\mathbf{a}']}$, because their corresponding generators are different. Therefore, since $(I^{[\mathbf{a}_I]})^{[\mathbf{a}_I]} = I$, it follows that $(I^\vee)^\vee = I$ if and only if $\mathbf{a}_I = \mathbf{a}_{I^\vee}$.

Claim B.1 $\mathbf{a}_I = \mathbf{a}_{I^\vee}$ *if and only if for each* $i \in \operatorname{supp}(\mathbf{a}_I)$ *there is a minimal generator* $\mathbf{x}^\mathbf{b}$ *of* I *with* $b_i = 1$.

Proof: $\Longrightarrow$: Let a_i and $b_i^\vee$ be the i^{th} coordinates of $\mathbf{a}_I$ and $\mathbf{a}_I \smallsetminus \mathbf{b}$. If there is some $\mathbf{x}^\mathbf{b}$ with $b_i = 1$, then $b_i^\vee = a_i$ by definition, so there is an irreducible component of $I^\vee$ with $x_i^{a_i}$ as one of its minimal generators. It follows from the algorithm in Section 0.6 for computing irreducible decompositions that a_i is the exponent on x_i in some minimal generator of $I^\vee$.

$\Longleftarrow$: If $x_i^{a_i}$ is the power of x_i in some minimal generator of $I^\vee$, then x_i is a minimal generator of some irreducible component of I, because $(I^\vee)^{[\mathbf{a}_I]}$ always equals I (Corollary 6.15). As before, this implies that x_i appears with exponent 1 in some minimal generator of I. □

Solution B.3.2

This follows from the definitions, using the following two observations.

1. $\operatorname{supp}(\mathbf{a} \smallsetminus \mathbf{b}) = \mathbf{1} \smallsetminus \operatorname{supp}(\mathbf{b}) = \operatorname{supp}(\mathbf{b})$.
2. $\sqrt{\bigcap \mathfrak{m}^\mathbf{b}} = \bigcap \sqrt{\mathfrak{m}^\mathbf{b}} = \bigcap \mathfrak{m}^{\operatorname{supp}(\mathbf{b})}$.

Solution B.3.3 *By Robert Forkel*

The function `AlexDual2` below is an implementation of the one-step algorithm [23, Theorem 2.1]. It tends to work faster in computer algebra systems (such as `Macaulay 2` [18]) designed for calculating kernels of matrices. The remaining two functions `AlexDual3` and `AlexDual4` are straightforward, and work faster in programs (like CoCoA) which deal well with monomials. Two examples and speed tests are given at the end.

```
Define AlexDual2(...)
  I:=Minimalized(ARGV[1]);
  If Len(ARGV)=2 Then A:=ARGV[2]
  Else A:=Log(LCM(Gens(I)));
  End;
```

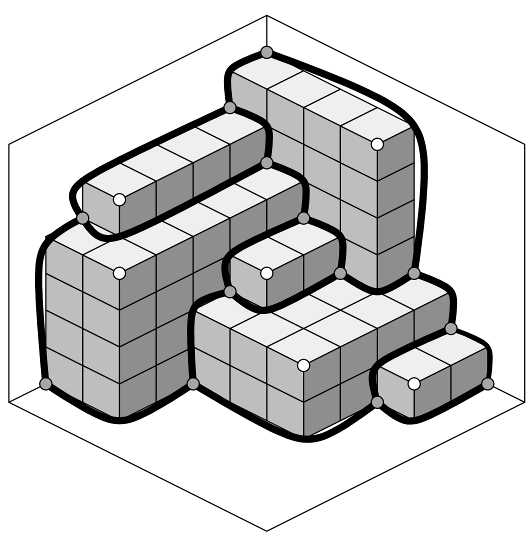
```
  If Len(ARGV)>2 Then Error('Too many arguments in AlexDual') End;
  M:=Ideal([Indet(N)^(1+A[N])|N In 1..Len(A)]);
  J:=Colon(M,I);
  Return(Ideal([X In Gens(J)|Max(Log(X)-A)<=0]));
End;-- AlexDual by Ezra Miller

Define AlexDual3(...)
  N:=NumIndets();
  I:=Minimalized(ARGV[1]);
  If Len(ARGV)=2 Then A:=ARGV[2]
  Else A:=Log(LCM(Gens(I)))
  End;
  I:=Mat([[1|K In 1..N]-Log(M)+A|M In Gens(ARGV[1])]);
  If Len(ARGV)>2 Then Error('Too many arguments in AlexDual') End;
  L:=[];
  For K:=1 To Len(I) Do
  Append(L,[(Indet(M))^I[K,M]|M In 1..N And I[K,M]<=A[M]]);
  End;
  Return IntersectionList([Ideal(M)|M In L])
End; -- AlexDual by Robert Forkel

Define AlexDual4(...)
  N:=NumIndets();
  I:=Minimalized(ARGV[1]);
  If Len(ARGV)=2 Then A:=ARGV[2]
  Else A:=Log(LCM(Gens(I))) End;
  I:=Mat([[1|K In 1..N]-Log(M)+A|M In Gens(ARGV[1])]);
  If Len(ARGV)>2 Then Error('Too many arguments in AlexDual') End;
  L:=[[Indet(M)^I[K,M]|M In 1..N And I[K,M]<=A[M]]|K In 1..Len(I)];
  Return IntersectionList([Ideal(M)|M In L])
End; -- AlexDual by Robert Forkel

Use S::=Q[abcd];
I:=Ideal(a^9, b^9, c^9, d^9, a^6b^7c^4d, a^2b^3c^8d^5, a^5b^8c^3d^2,
ab^4c^7d^6, a^8b^5c^2d^3, a^4bc^6d^7, a^7b^6cd^4, a^3b^2c^5d^8);
A2:=AlexDual2(I);
A3:=AlexDual3(I);
A4:=AlexDual4(I);
A2=A3 AND A3=A4;

Cpu time = 9.70, User time = 10
-------------------------------
Cpu time = 7.51, User time = 7
-------------------------------
```

```
Cpu time = 7.53, User time = 7
-------------------------------
TRUE
-------------------------------

Use S::=Q[uvtxyz];
I:=Ideal(u^8t^4xy^2z,tyz^3,x^2z^2,t^6z^5,v^5u^3xy,u^9,v^7xy);
A:=[10,10,10,10,10,10];
Set Timer;
A2:=AlexDual2(I);
A3:=AlexDual3(I);
A4:=AlexDual4(I);
A2=A3 AND A3=A4;

Cpu time = 6.88, User time = 7
-------------------------------
Cpu time = 3.00, User time = 3
-------------------------------
Cpu time = 3.01, User time = 3
-------------------------------
TRUE
-------------------------------
```

Solution B.3.4

Most of this exercise is a specific example of Proposition 5.4 and Example 6.8, where the polytope is triangle × triangle × segment. The vertices are labeled by the minimal generators of $M = \langle x_i y_j z_k \mid i, j \in \{0, 1, 2\} \text{ and } k \in \{0, 1\}\rangle$ by putting the label $x_i y_j z_k$ on the vertex $x_i \times y_j \times z_k$.

Solution B.3.5



```
-- Current ring is R = Q[t,x,y,z]
-------------------------------
M := Ideal(z^6, xz^5, xyz^4, xy^2z^3, y^4z^2, xy^3z^2, y^5z, y^6,
   x^4y^2, x^6, x^3y^2z^2, x^2y^5, x^5z^4);
Res(M);
0 --> R^2(-9)(+)R^3(-11)(+)R(-12) -->
R^7(-7)(+)R^6(-8)(+)R(-9)(+)R^4(-10) --> R^10(-6)(+)R^2(-7)(+)R(-9)
-------------------------------
```

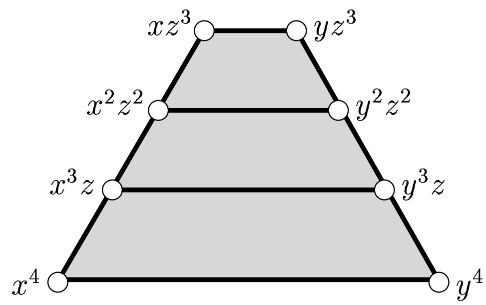
Thus CoCoA agrees with the picture, since there are $2+3+1=6$ regions, $7+6+1+4=18$ edges, and $10+2+1=13$ dark vertices.

Solution B.3.6 *By Moira McDermott, Hal Schenck, Greg Smith, and Carolyn Yackel**

Use Alexander duality: the maximal number of minimal generators of an intersection of 12 irreducible ideals in $k[x_1,x_2,x_3,x_4]$ is the same as the maximal number of irreducible components of an ideal generated by 12 monomials in $k[x_1,x_2,x_3,x_4]$. Moreover, the generic ideal in Example 4.15 attains the bound of $53 = C_{3,4,12}-1$. The "-1" is because the Scarf complex of a generic artinian ideal is obtained from a polytope by leaving off at least one face.

Solution B.3.7

The hull resolution can be computed by hand to be



and yields a minimal resolution because none of the faces have the same label. The irreducible decomposition

$$\langle x^4,y^4,z\rangle \cap \langle x^3,y^3,z^2\rangle \cap \langle x^2,y^2,z^3\rangle \cap \langle x,y\rangle$$

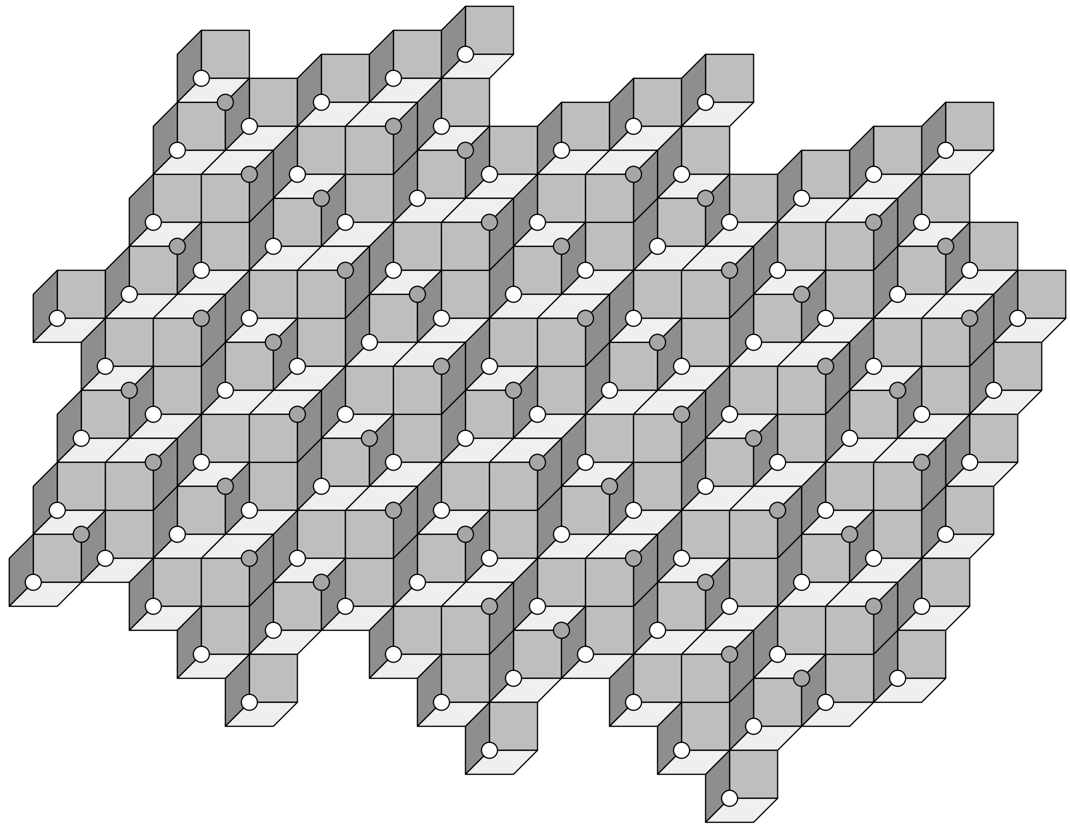
has three terms appearing on the facets of the hull complex. The last component $\langle x,y\rangle$ "wants" to appear in the truncated top corner of the triangle, but fails to because it isn't artinian. Throwing in an extra generator z^4 to the ideal makes this last component appear.

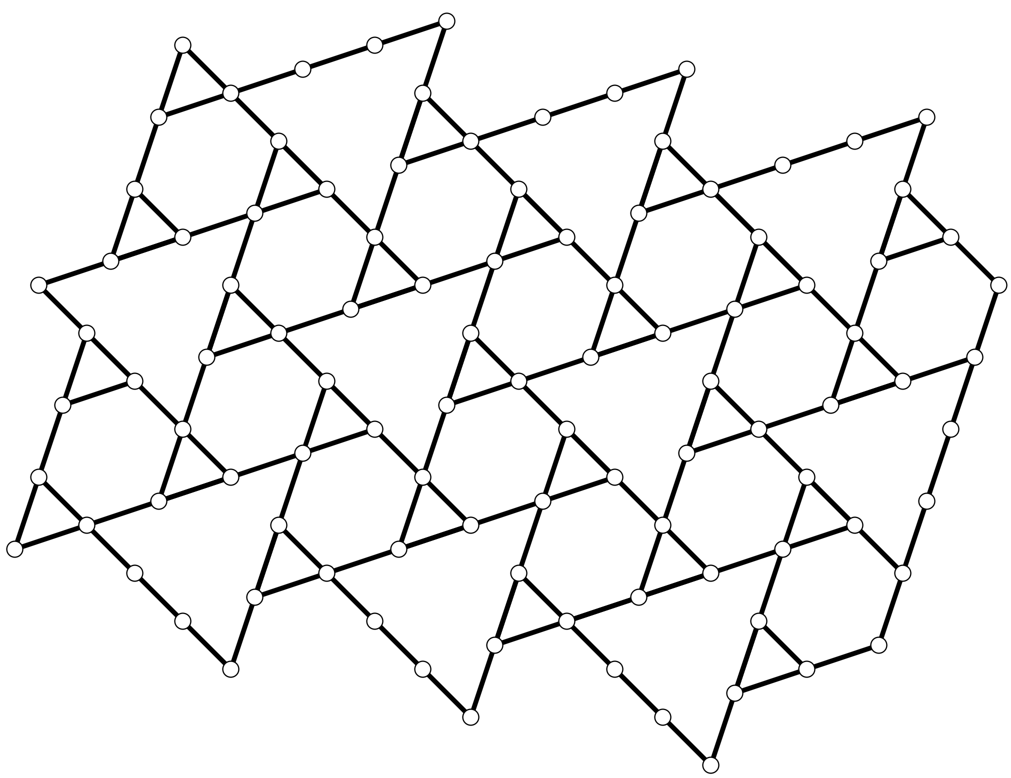
Solution B.3.8

The hull complex is a truncated 4-simplex which is the convex hull of the exponent vectors of I. For any squarefree ideal I whose generators all have the same $\mathbb{Z}$-degree, the hull complex is the convex hull of the exponents on the minimal generators; see [4, Corollary 2.13].

B.4 Solutions for Day 4

Solution B.4.1

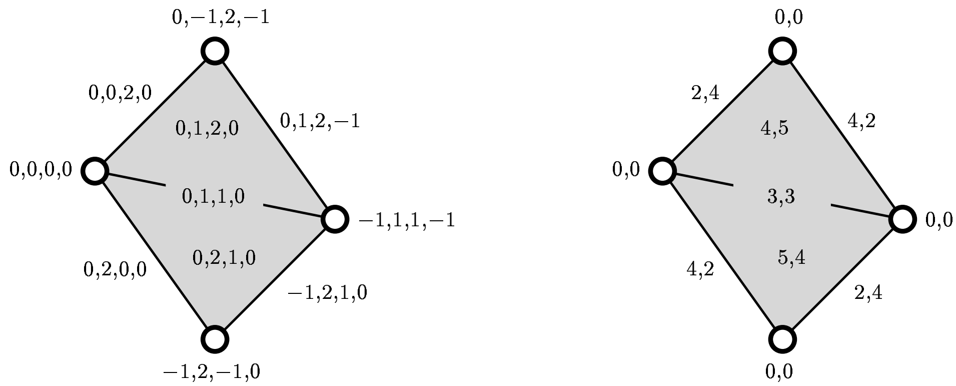




The hull resolution is not minimal: hull(M_L) is obtained from the cell complex above by introducing three short edges into each large "triangle", leaving only hexagons and small triangles. The hull resolution can be made minimal by removing any one of the three edges from each resulting small "down" triangle, but we have chosen the method in the diagram for aesthetic reasons.

Solution B.4.2

The hull complex of M_L consists of shifts of the up and down triangles below.



Since the labels are all distinct, the resolution is minimal. Notice how the opposing edge labels naturally become identified when the matrix defining L is applied to them.

Solution B.4.3

The canonical module ω is spanned as a k-vector subspace of $k(t, t^{-1})$ by the monomials $t^{-2}, t^{-1}, t, t^2, t^3, \ldots$ (with $t^0 = 1$ missing). These powers of t are the inverses of the powers of t that aren't in $k[t^3, t^4, t^5]$; see [7, Corollary 4.3.8]. Therefore, ω is the image under the functor π of the monomial module M_ω which is k-spanned by the *inverses* of the Laurent monomials *not* in M_L, where $L = \ker([4\ 3\ 5])$. What does this have to do with Alexander duality? A picture of the monomial module M_ω is obtained from the picture of M_L in Example 7.10 by turning the page over. This is the same instruction you were supposed to follow to get the Alexander dual ideal from the original in Example 6.12. And remember the definition of Alexander dual simplicial complex: $\Delta^\vee$ consists of the *complements* of faces that are *not* in Δ. For more on these connections, see [23].

Solution B.4.4 *By Bahman Engheta, Leah Gold*, and Ed Mosteig*

Finding the Hilbert series of $k[t^{20}, t^{24}, t^{25}, t^{31}]$ is equivalent to finding the Hilbert series of the isomorphic ring $Q[a, b, c, d]/I_L$, where L is the lattice generated by the kernel of the matrix [20 24 25 31] and the variables have degrees 20, 24, 25, and 31. The following CoCoA routine calculates the Hilbert series of $Q[a, b, c, d]/I_L$.

```
Use S ::= Q[t,x,y,z], Weights(20,24,25,31);
Poincare(S/Toric([[20,24,25,31]]));
```

We find that the numerator $p(t)$ is

$$\begin{aligned}&-t^{183}-t^{178}-t^{177}-t^{167}-t^{166}-t^{161}\\&+t^{158}+t^{153}+t^{152}+t^{147}+t^{146}+t^{143}+t^{142}+t^{141}+t^{137}+t^{136}+t^{135}+t^{130}\\&-t^{122}-t^{112}-t^{110}-t^{96}-t^{93}-t^{80}-t^{75}+1\,.\end{aligned}$$

Each term t^d appearing in the above polynomial corresponds to a face of hull(I_L). The exponent d gives the degree of the face in $\mathbb{Z}^4/L \cong \mathbb{Z}$. Hence there are 7 edges with degrees 75, 80, 93, 96, 110, 112, and 122; 12 faces of dimension 2 with degrees 130, 135, 136, 137, 141, 142, 143, 146, 147, 152, 153, and 158; and 6 faces of dimension 3 with degrees 161, 166, 167, 177, 178, and 183.

Solution B.4.5 *By Bahman Engheta, Leah Gold*, and Ed Mosteig*

Let $S = Q[t^{20}, t^{24}, t^{25}, t^{31}]$. Then to find the largest amount of money which cannot be expressed by the 4 coins, we simply need to find the largest d for which t^d is not in S. That is, find the largest d such that $\dim_k(S_d) = 0$. The Hilbert series for S, which we found in the previous Solution can be written as

$$\begin{aligned}&1+t^{20}+t^{24}+t^{25}+t^{31}+t^{40}+t^{44}+t^{45}+t^{48}+t^{49}+t^{50}+t^{51}+t^{55}+t^{56}+t^{60}+t^{62}\\&+t^{64}+t^{65}+t^{68}+t^{69}+t^{70}+t^{71}+t^{72}+t^{73}+t^{74}+t^{75}+t^{76}+t^{79}+t^{80}+t^{81}+t^{82}\\&+\textstyle\sum_{d=84}^{\infty} t^d\,.\end{aligned}$$

As we know, the coefficient of t^d in the Hilbert series is $\dim_k(S_d)$. Hence, the largest amount of money which cannot be expressed by the 4 coins is 83 units. This number is most simply calculated as the degree of the Hilbert series as a rational function—that is, the degree of the numerator (= 183) minus the sum of the degrees of the variables (= 100).

Solution B.4.6

This exercise is analogous to, but more complicated than, Exercise A.2.5. Since L is generic, the hull complex is the Scarf complex, which makes things easier. Here's the problem: although the terms in the numerator of the $\mathbb{Z}^n/L$-graded Hilbert series of S/I_L are in bijection with the faces of hull(I_L) for generic L, it's harder to get the partial order information on the faces from the same source. Therefore, we prefer to get the partial order on the face labels before quotienting hull(M_L) by L. We will accomplish this by choosing a `WeightsMatrix` which records the finer $\mathbb{Z}^n$-grading in a coherent way.

First things first: let's find generators for I_L. We need to have a positive functional which vanishes on L, since CoCoA requires homogeneity with respect to the first row of the `WeightsMatrix` in order to apply the operator `Toric`.

```
M := Mat([[-7,-5,3,8],[4,-7,9,-1]]);
K := LinKer(M);
K;
[[-3, 0, 1, -3], [-5, 25, 22, 3]]
```

```
-------------------------------
K[2] - 2K[1];
[1, 25, 20, 9]
-------------------------------
```

As in Solution B.2.5, we now define our ring with one dummy variable `t` to make things homogeneous. The rest of the `WeightsMatrix` keeps track of the $\mathbb{Z}^n$-grading. For now, the ideal `J` is defined by the binomials coming directly from the two row vectors of `M`.

```
Use S ::= Q[t,a,b,c,d], Weights(Mat([[1,1,25,20,9],
                                     [0,1,0,0,0],
                                     [0,0,1,0,0],
                                     [0,0,0,1,0],
                                     [0,0,0,0,1]]));
Pos(V) :=  1/2(V + [Abs(X) | X In V]);
Neg(V) := -1/2(V - [Abs(X) | X In V]);
J := Ideal([LogToTerm(Concat([0],Pos(X)))
          - LogToTerm(Concat([0],Neg(X))) | X In M]);
J;
Ideal(-a^7b^5 + c^3d^8, a^4c^9 - b^7d)
-------------------------------
```

The lattice ideal `I` $= I_L$ contains `J`, and is obtained from `J` by saturation with respect to the product of all the variables. CoCoA has a new built-in command to do this:

```
I := Toric(Gens(J));
I;
Ideal(-a^7b^5 + c^3d^8, a^4c^9 - b^7d, a^11c^6 - b^2d^9, -a^18b^3c^3 +
d^17, -a^3b^12 + c^12d^7, -b^19 + ac^21d^6)
-------------------------------
```

The faces of $\text{hull}(I_L) = \text{hull}(M_L)/L$ are represented once each in the $\mathbb{Z}^n/L$-graded Hilbert series of S/I_L, and our `Weights` are set up to give us $\mathbb{Z}^n$-graded labels for them, listed in `Scarf` (along with the corresponding ± 1 coefficient in the Hilbert series).

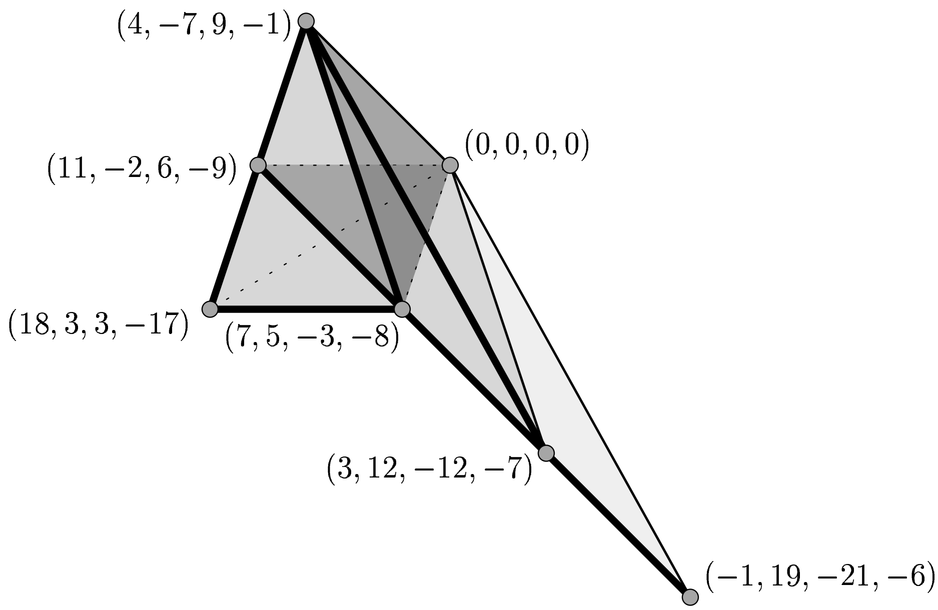
```
P := Poincare(S/I);
P;
---  Non-simplified HilbertPoincare' Series  ---
( - t^487a^7b^12c^9 + t^484a^4b^12c^9 + t^478a^3b^19 - t^475b^19 -
t^316a^11b^5c^9 + t^312a^7b^5c^9 + t^307a^7b^12 - t^303a^3b^12 -
t^263a^18b^5c^6 + t^256a^11b^5c^6 + t^213a^18b^3c^6 +
t^203a^18b^5c^3 + t^191a^11c^9 - t^184a^4c^9 - t^153a^18b^3c^3 -
t^131a^11c^6 - t^132a^7b^5 + 1) / ( (1-t) (1-ta) (1-t^25b) (1-t^20c)
(1-t^9d) )
-------------------------------
```

```
Scarf := [[X[1],Tail(X[2])] | X In P[1]];
Scarf;
[[-1, [7, 12, 9, 0]], [ 1, [4, 12, 9, 0]], [ 1, [3, 19, 0, 0]],
 [-1, [0, 19, 0, 0]], [-1, [11, 5, 9, 0]], [ 1, [7,  5, 9, 0]],
 [ 1, [7, 12, 0, 0]], [-1, [3, 12, 0, 0]], [-1, [18, 5, 6, 0]],
 [ 1, [11, 5, 6, 0]], [ 1, [18, 3, 6, 0]], [ 1, [18, 5, 3, 0]],
 [ 1, [11, 0, 9, 0]], [-1, [4,  0, 9, 0]], [-1, [18, 3, 3, 0]],
 [-1, [11, 0, 6, 0]], [-1, [7,  5, 0, 0]], [ 1, [ 0, 0, 0, 0]]]
-------------------------------
```

Hopefully the lifts to $\mathbb{Z}^n$ we have chosen for the labels are coherent enough that the partial ordering on them is the desired Scarf complex. For each face `X` of hull(I_L), `Leq` picks out the other faces whose labels divide the label on `X` and whose dimension is of opposite parity (this is just a convenience which avoids listing too many faces).

```
Divides(Y,X) := Mod(LogToTerm(Y),LogToTerm(X)) = 0;
Leq(X,Scarf) := [Y[2] | Y In Scarf And Divides(X[2],Y[2])
                        And X[1] = -Y[1] And Y[2] <> [0,0,0,0]];
PartOrder := [[X[2],Leq(X,Scarf)] | X In Scarf];
PartOrder;
[[[7, 12, 9, 0], [[4, 12, 9, 0], [7, 5,  9, 0], [7, 12, 0, 0]]],
 [[4, 12, 9, 0], [[3, 12, 0, 0], [4, 0,  9, 0]]],
 [[3, 19, 0, 0], [[0, 19, 0, 0], [3, 12, 0, 0]]],
 [[0, 19, 0, 0], [ ]],
 [[11, 5, 9, 0], [[7, 5,  9, 0], [11, 5, 6, 0], [11, 0, 9, 0]]],
 [[7,  5, 9, 0], [[4, 0,  9, 0], [7,  5, 0, 0]]],
 [[7, 12, 0, 0], [[3, 12, 0, 0], [7,  5, 0, 0]]],
 [[3, 12, 0, 0], [ ]],
 [[18, 5, 6, 0], [[11, 5, 6, 0], [18, 3, 6, 0], [18, 5, 3, 0]]],
 [[11, 5, 6, 0], [[11, 0, 6, 0], [7,  5, 0, 0]]],
 [[18, 3, 6, 0], [[18, 3, 3, 0], [11, 0, 6, 0]]],
 [[18, 5, 3, 0], [[18, 3, 3, 0], [7,  5, 0, 0]]],
 [[11, 0, 9, 0], [[4,  0, 9, 0], [11, 0, 6, 0]]],
 [[4,  0, 9, 0], [ ]],
 [[18, 3, 3, 0], [ ]],
 [[11, 0, 6, 0], [ ]],
 [[7,  5, 0, 0], [ ]],
 [[0,  0, 0, 0], [ ]]]
-------------------------------
```

This information is enough to draw a picture, although we will see that there's still some partial ordering left to do.



If B is the simplicial complex outlined in bold and containing {3 triangles plus an edge}, then the diagram is supposed to look like the cone over B. The complex hull(I_L) is obtained by removing B from the above cone, and consists of 3 tetrahedra, 8 triangles, 6 edges, and 1 vertex; check `Res(S/I)` for verification. Each face of B is identified in hull(I_L) with a face appearing elsewhere in the diagram. For instance,

$$\{(4,-7,9,-1),(7,5,-3,-8),(3,12,-12,-7)\} \equiv \{(0,0,0,0),(3,12,-12,-7),(-1,19,-21,-6)\},$$

is an equivalence modulo L, as is

$$\{(11,-2,6,-9),(18,3,3,-17),(7,5,-3,-8)\} \equiv \{(4,-7,9,-1),(11,-2,6,-9),(0,0,0,0)\}.$$

The labels in `PartOrder` are the least common multiples of the labels on the vertices.

You may notice something strange about `PartOrder`: it not quite complete. In particular, each tetrahedron seems to have only 3 facets, and each triangle only 2 facets. The $\mathbb{Z}^n$-grading has "forgotten" that the triangle

$$\{(0,0,0,0),(3,12,-12,-7),(-1,19,-21,-6)\}$$

is a face of the tetrahedron

$$\{(0,0,0,0),(4,-7,9,-1),(7,5,-3,-8),(3,12,-12,-7)\}$$

after we quotient by L. In general, we have to add to the partial order the relations determined by the fact that every face in hull(I_L) has one of its boundary faces in B. See [30] for details.

Question for further thought: Were we lucky that the lifting of the $\mathbb{Z}^n/L$-grading gave us a partially ordered set that was already a simplicial complex, without having to fill in more faces, or is this a general phenomenon? What choices did we make (implicitly or otherwise) that made things come out right? To understand the question better, look what happens to the last diagram in Example 7.10 if we choose the $\mathbb{Z}^n$-graded lift $(1,0,1)$ for the interior edge instead of $(0,3,0)$.

Solution B.4.7 See Example 7.17; we do not give details here.

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