Math 111: Calculus

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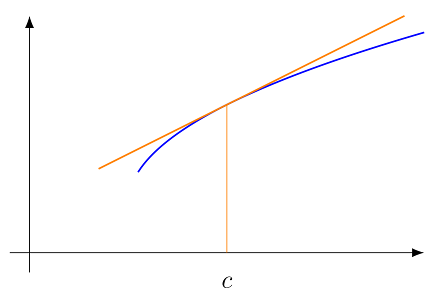
Contents

Week 1, Monday: Introduction: derivatives, integrals, and the fundamental theorem.

Overview of Calculus

The main idea of calculus is to approximate curvy things with straight things. It applies this idea to two seemingly unrelated topics: rates of change and areas.

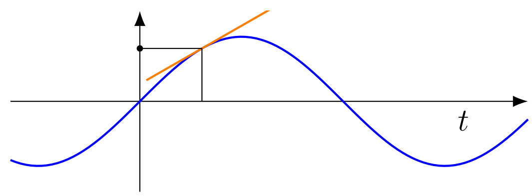
Derivatives. Let g be a function: for each real number x, the function gives a corresponding number $g(x)$. At each real number c, calculus finds the "best" line approximating the function:



Graph of a function g and its best linear approximation at the point c.

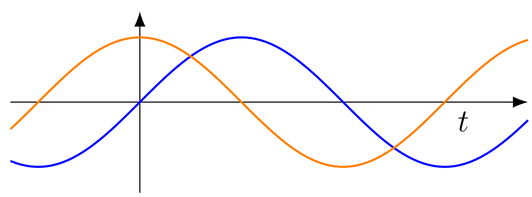
The slope of the best approximating line at a point c is the *rate of change* or *slope* of the function at that point. In the graph above, you can see that the slope of g changes from point to point: it is decreasing as c increases.

If you think of $g(t)$ as specifying the distance of a point on the y-axis from the origin at time t, then the derivative of g at time $t = c$, denoted $g'(c)$, is the *speed* the particle is traveling. For instance, suppose $g(t) = \sin(t)$. The graph appears below:



Graph of $g(t) = \sin(t)$.

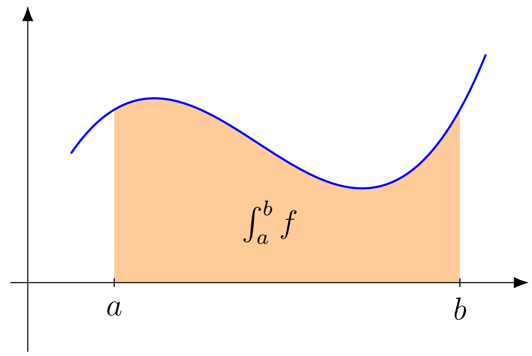
Imagine what happens to the point on the y-axis as time moves forward: it oscillates between -1 and 1. At the point pictured above, the derivative of g, i.e., the speed of the particle, is positive—the point is moving away from the origin. The derivative of $\sin(t)$ happens to be $\cos(t)$. The next picture shows the graphs of these functions superimposed.



Graph of $\sin(t)$ in blue and its derivative $\cos(t)$ in orange.

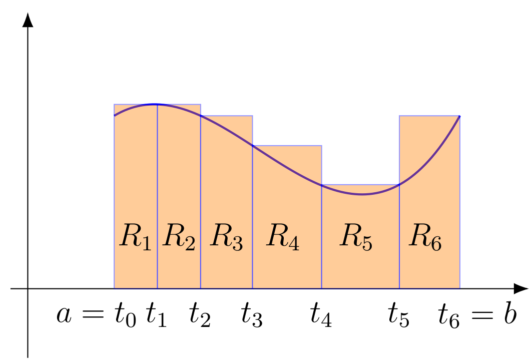
The height of the orange graph gives the slope of the blue graph at each point. For instance, notice that where the orange graph is 0 (where it hits the x-axis) the blue graph flattens out. Where the orange graph is positive, the blue graph is increasing—it's slope is positive, and where the orange graph is negative, the blue graph is decreasing.

Integrals. The integral of a function $f(t)$ from $t = a$ to $t = b$ is the area under the graph of f between those two points:



The integral $\int_a^b f$ is the area under the curve from $t = a$ to $t = b$.

The approach of calculus to find this area is to first estimate it using rectangles (replacing curvy things by straight things):

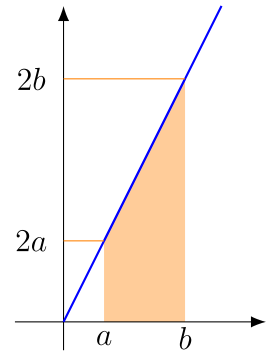


Approximating the area under the graph of f with rectangles.

The Fundamental Theorem of Calculus. It turns out there is an essential connection between derivatives and integrals. To fix ideas, let g be a distance function for a particle on the y-axis, and at each time t, let $f(t)$ be the speed of the particle. So $g'(t) = f(t)$. The Fundamental Theorem of Calculus says the following:

$$\underbrace{\int_a^b f}_{\text{area under } f \text{ from } a \text{ to } b} = \int_a^b g' = \underbrace{g(b) - g(a)}_{\text{net change in } g}.$$

Example of the Fundamental Theorem. Let $g(t) = t^2$. It turns out that $g'(t) = 2t$. Then $\int_a^b g'$ is the area pictured below:



$\int_a^b g'$ is the area under the graph of $g'(a)$ from a to b.

The area under $g')$ from a to b can be calculated by subtracting the areas of two triangles: the one from the origin out to the line at $t = b$ minus the one from the origin out to the line $t = a$:

$$\int_a^b g' = \frac{1}{2} b(2b) - \frac{1}{2} a(2a) = b^2 - a^2.$$

Recall that $g(t) = t^2$. So in accordance with the Fundamental Theorem of Calculus, we have

$$\int_a^b g' = g(b) - g(a).$$

Summary of goals for Math 111

- What is speed? (derivatives)
- What is area? (integrals)
- How are they related? (Fundamental Theorem of Calculus (FTC))
- Theory:
  - IVT (intermediate value theorem)
  - EVT (extreme value theorem)
  - MVT (mean value theorem)
  - Chain rule, product rule

- FTC.

- Applications:
  - Calculate speed and area efficiently.
  - Optimization (maximize and minimize functions).
  - Related rates.
  - Differential equations and population models.

The first technical definition we'll need to come to terms with is the following (to be taken up during the next lecture):

Definition. Let f be a function defined in an open interval containing a point c, except f might not be defined at the point c, itself. Let L be a real number. The *limit of $f(x)$ as x approaches c* is L, denoted $\lim_{x \to c} f(x) = L$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

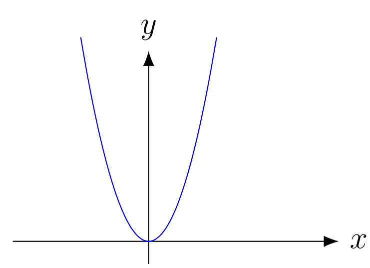
$$0 < |x - c| < \delta$$

implies

$$|f(x) - L| < \varepsilon.$$

Week 1, Wednesday: Average speed, instantaneous speed. Definition of the limit.

Slopes. Let $f(t) = t^2$ be a function describing the position of a particle on the real number line at time t. For instance, at time $t = 2$, the particle is at $f(2) = 4$.



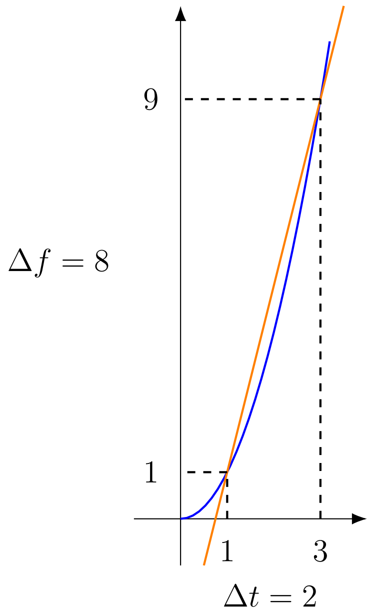
Graph of $f(t) = t^2$.

Problem: Find the average speed of the particle from time $t = 1$ to time $t = 3$.

Solution: The total distance traveled is $f(3) - f(1) = 3^2 - 1^2 = 8$, and the time elapsed is $3 - 1 = 2$. So

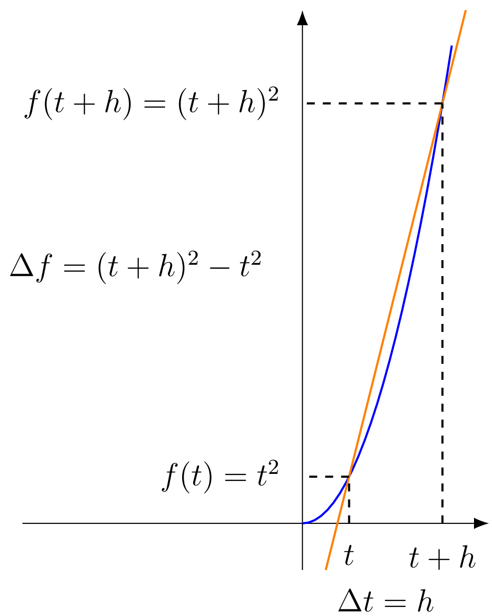
$$\text{average speed} = \frac{\Delta f}{\Delta t} = \frac{f(3) - f(1)}{3 - 1} = \frac{8}{2} = 4.$$

Relevant picture:



Graph of $f(t) = t^2$ and a secant line.

Problem: Find the average speed from an arbitrary time t to time $t+h$ for some $h > 0$.



Graph of $f(t) = t^2$ and a secant line.

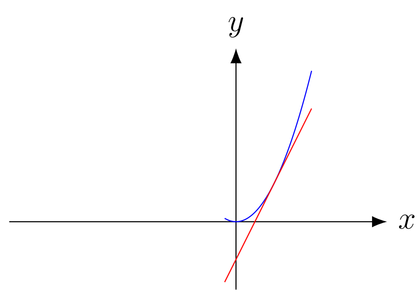
We have

$$\begin{aligned}\text{average speed} &= \frac{\Delta f}{\Delta t}\\ &= \frac{(t+h)^2 - t^2}{(t+h)-t}\\ &= \frac{t^2+2th+h^2-t^2}{h}\\ &= \frac{2th+h^2}{h}\\ &= \frac{(2t+h)h}{h}\\ &= 2t+h.\end{aligned}$$

(In the last step, we used the fact that $h \neq 0$.) For example, the average speed from time 1 to time $1.1 = 1 + 0.1$ is

$$2 \cdot 1 + 0.1 = 2.1.$$

Let h get really small to estimate the *instantaneous* speed at time t: as the time interval $h = (t+h) - t$ approaches 0, the average speed approaches $2t$. We've just calculated the *derivative*, $f'(t)$, of f at an arbitrary time t and found that $f'(t) = 2t$.



Graph of $f(t) = t^2$ with attached tangent line.

Limits. Above, we just considered the behavior of the quotient

$$\frac{\Delta f}{\Delta t} = \frac{f(t+h) - f(t)}{h}$$

as the time interval, h, tends to 0, and we thought of t as begin fixed. So, we are really considering a function of h:

$$g(h) = \frac{f(t+h) - f(t)}{h}.$$

The interesting thing about this function $g(h)$ is that it is *not defined* at the point we are interested in, i.e., at $h = 0$. This will *always* happen when we are interested in calculating an instantaneous speed. Our task, though, is to find out what $g(h)$ gets close to as h gets close to 0.

Problem: Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x = 1/10^n \text{ for } n = 1, 2, \ldots \\ 3 & \text{otherwise.} \end{cases}$$

Is it true that $f(x)$ gets close to 3 as x gets close to 0? Is it true that $\lim_{x\to 0} f(x) = 0$?

Problem: Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Is it true that as $f(x)$ gets close to 1 as x get close to 0? Gets close to -1? Is it true that $\lim_{x\to 0} f(x) = 1$ or $\lim_{x\to 0} f(x) = -1$.

The following definition precisely captures the idea we need, no more and no less.

Definition. Let f be a function defined in an open interval containing a point c, except f might not be defined at the point c, itself. Let L be a real number. The *limit of* $f(x)$ *as* x *approaches* c is L, denoted $\lim_{x \to c} f(x) = L$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta$$

implies

$$|f(x) - L| < \varepsilon.$$

Week 1, Friday: Limits.

Definition. Let f be a function defined in an open interval containing a point c, except f might not be defined at the point c, itself. Let L be a real number. The *limit of* $f(x)$ *as* x *approaches* c is L, denoted $\lim_{x\to c} f(x) = L$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta$$

implies

$$|f(x) - L| < \varepsilon.$$

Comments.

- The definition of the limit contains a huge amount of information. Unless you have worked with it before—which I am not assuming—don't expect to understand it on first reading (or on the second or third, for that matter).

- We are interested in the behavior of the function f near the point c, but not exactly at the point c. In fact, f need not even be defined at c. For example, consider the function

  $$f(x) = \frac{x^2 - x}{x}.$$

  If we try to evaluate f at 0, we get $f(0) = \frac{0}{0}$, which does not makes sense (you can't divide by 0), i.e., $\frac{0}{0}$ is not a number.

- When you see the absolute values in the definition, you should think "distance". The *distance* between real numbers a and b is $|a - b|$. So you should translate $|f(x) - L| < \varepsilon$ as "the distance between $f(x)$ and the number L is less than ε".

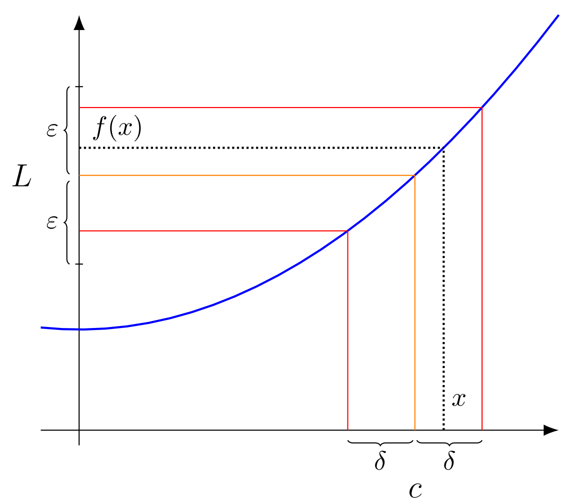
- Consider the part of the definition that says $0 < |x - c| < \delta$. If the expression had just been $|x - c| < \delta$, the requirement would be that the distance between the number x and c is less than δ. What about the fact the $0 < |x - c|$? The only way the absolute value of a number such as $x - c$ can be 0 is if the number itself is 0, i.e., $x - c = 0$ or, equivalently, $x = c$. Thus, requiring $0 < |x - c|$ is just requiring that x not equal c. This is just what we need since, after all, the function f may not be defined at c.

- Note the *quantifiers* "for all" and "there exists" in the definition. It takes a while to appreciate their importance, but they are essential. First take the "for all" part. The definition say that for all $\varepsilon > 0$, we are going to want $|f(x) - L| < \varepsilon$. Translating: for all $\varepsilon > 0$, we will want to make the distance between $f(x)$ and L less that $\varepsilon > 0$. Our goal is to make f close to L, and the ε is a measure of how close. By making ε small and requiring $|f(x) - L| < \varepsilon$, we are ensuring that $f(x)$ is within a distance of ε from L.

  Next, consider the "there exists" part of the definition. It says that if you want $f(x)$ to be with a distance of ε of L, then you need to make $0 < |x - c| < \delta$. In other words, you need to make x within a distance of δ of c (remembering that we don't care what happens when $x = c$).

  Given any $\varepsilon > 0$ (a challenge to make $f(x)$ close to L), you want to find an appropriate distance $\delta > 0$ (so that if x is δ-close to c, then $f(x)$ is ε-close to L).

- The relevant picture:

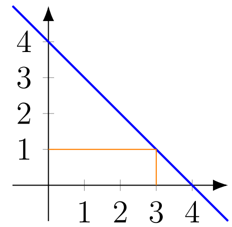


Meaning of ε and δ in the definition of the limit.

If f is steep near c, then δ needs to be taken smaller.

Problems: Consider some problems from page 55 of the text.

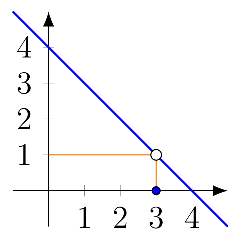
9. $\lim_{x \to 3} 4 - x = 1$.



11.

$$f(x) = \begin{cases} 4 - x & \text{if } x \neq 3 \\ 0 & \text{if } x = 3. \end{cases}$$

Here, $\lim_{x \to 3} = 1$, again. The limit would be the same even if f were not defined at all at $x = 3$.

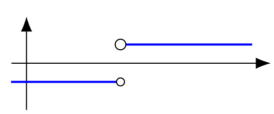


13.

$$f(x) = \frac{|x - 5]}{x - 5}.$$

Here, f is not defined at $x = 5$. However, in fact, we have

$$f(x) = \frac{|x - 5]}{x - 5} = \begin{cases} 1 & \text{if } x > 5 \\ -1 & \text{if } x < 5 \\ \text{undefined} & \text{if } x = 5. \end{cases}$$



In this case, f does not have a limit as x approaches 5. This makes sense: if x approaches 5 from numbers greater than 5, then $f(x)$ gets close to 1 (and in fact is exactly equal to 1 for all numbers greater than 5). However, the limit can't be 1 since there will be numbers x, equally close to 5 but to the left of 5 for which $f(x) = -1$. (Here, there is no way we will be able to "beat" and ε that is less than or equal to 2 since requiring $|f(x) - L| < 2$, i.e., strictly less than 2, will be bad news if f varies from 1 to -1 when close to $x = 5$.

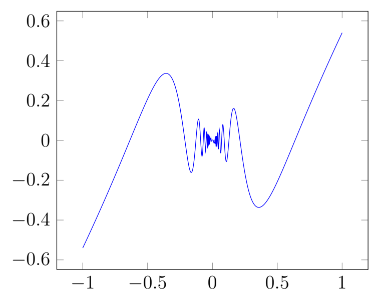
Problem. Prove that $\lim_{x \to 3} 2x + 5 = 11$.

Proof. Given $\varepsilon > 0$, let $\delta = \varepsilon/2$. Suppose that $0 < |x - 3| < \delta$; in other words, suppose that $0 < |x - 3| < \varepsilon/2$. Then

$$\begin{aligned} |(2x + 5) - 11| &= |2x - 6| \\ &= |2(x - 3)| \\ &= 2|x - 3| \\ &< 2 \cdot \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

□

Problem. Prove that $\lim_{x \to 0} x \cos(1/x)$.



Graph of $f(x) = x \cos(1/x)$.

Proof. Given $\varepsilon > 0$, let $\delta = \varepsilon$. Suppose that $0 < |x - 0| < \delta$; in other words, suppose that $0 < |x| < \varepsilon$. Then, since $|\cos(y)| \le 1$ for all y, we have

$$\begin{aligned} |x \cos(1/x)| &= |x||\cos(1/x)| \\ &< |x| \\ &= \varepsilon. \end{aligned}$$

□

Week 2, Wednesday: Limit theorems.

Please refer to last Friday's lecture. I'll start by adding the picture illustrating the relation between ε and δ that we didn't cover in class on Friday.

Examples of limits.

Claim. $\lim_{x\to 7} 5x - 4 = 31$.

Proof. Given $\varepsilon > 0$, let $\delta = \varepsilon/5$. Suppose that $0 < |x-7| < \delta = \varepsilon/5$. Then

$$|(5x-4)-31| = |5x-35| = 5|x-7| < 5\cdot\frac{\varepsilon}{5} = \varepsilon,$$

as required. $\square$

Suppose we have found one $\delta > 0$ such that $0 < |x-c| < \delta$ implies $|f(x)-L| < \varepsilon$. Then take $\delta' > 0$ such that $\delta' < \delta$. It follows that if x satisfies $0 < |x-c| < \delta'$, then $0 < |x-c| < \delta$, too, and hence, $|f(c)-L| < \varepsilon$. The point here is that "once you've found a suitable δ, you can always make δ smaller". We use that fact in the following proof.

Claim. $\lim_{x\to 1} 6 - 1/x = 5$.

Proof. Given $\varepsilon > 0$, let $\delta = 0.5$ and suppose that $0 < |x-1| < \delta = 0.5$. We have

$$|(6-1/x)-5)| = |1-1/x| = \left|\frac{x-1}{x}\right| = |x-1|\cdot\frac{1}{|x|}.$$

Since $|x-1| < \delta = 0.5$, it follows that $0.5 < x < 1.5$. In particular, since $0.5 < x$, we have $1/|x| > 1/0.5 = 2$. Therefore,

$$|(6-1/x)-5)| = |x-1|\cdot\frac{1}{|x|} < 2|x-1|.$$

Now replace δ by the minimum of $\varepsilon/2$ and 0.5, whichever is smallest. Suppose that $0 < |x-1| < \delta$. Then, since $\delta \le 0.5$, we still have that $1/|x| < 2$, and thus $|(6-1/x)-5| < 2|x-1|$. In addition, since $\delta \le \varepsilon/2$, we have

$$|(6-1/x)-5| < 2|x-1| < 2\cdot\frac{\varepsilon}{2} = \varepsilon,$$

as required. $\square$

For instance, suppose we want to make the function $f(x) = 6 - 1/x$ within a distance of 0.1 of 5 by making x close to 1. Using the above proof with $\varepsilon = 0.1$, we see that we can take any $\delta > 0$ satisying $\delta \leq 0.5$ and $\delta \leq \varepsilon/2 = 0.1/2 = 0.05$. Thus, for any x satisfying $0 < |x - 1| < 0.05$, we have $|(6 - 1/x) - 5| < 0.1$.

The proof of next example is similar to that just seen. Note the simplicity of the function $f(x) = x^2$ and the seeming obviousness of the claim compared to difficulty of the proof!

Claim. $\lim_{x \to 5} x^2 = 25$.

Proof. Given $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/11\}$, i.e., δ is the minumum of 1 and $\varepsilon/11$. So $\delta \leq 1$ and $\delta \leq \varepsilon/11$ (with equality holding in at least one of these). Suppose that x satisfies $0 < |x - 5| < \delta$. Since $\delta \leq 1$, it follows $4 < x < 6$, and hence $9 < x + 5 < 11$. In particular, $|x + 5| < 11$. Therefore,

$$|x^2 - 25| = |(x + 5)(x - 5)| = |x + 5||x - 5| < 11|x - 5|.$$

Now, since $\delta \leq \varepsilon/11$ and $|x - 5| < \delta$, it follows that

$$|x^2 - 25| < 11|x - 5| < 11 \cdot \frac{\varepsilon}{11} = \varepsilon,$$

as required. $\square$

So even with very simple functions like $f(x) = x^2$, these limit proofs are difficult. What is one to do? The answer is the following very general limit theorem:

Limit Theorem. Suppose that $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist. Then

1. $\lim_{x \to c}(f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$.
2. $\lim_{x \to c} f(x)g(x) = \lim_{x \to c} f(x) \lim_{x \to c} g(x)$.
3. If $\lim_{x \to c} g(x) \neq 0$, then
$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}.$$

In words, this theorem says that if we know the limits of f and g, then we can easily find the limit of any function we can construct from f and g using the operations of addition, multiplication, and division. For instance, combined with the following proposition, we can easily find the limit of any quotient of polynomials. We'll see that next time, but for a preview let's revisit a result we proved earlier:

Claim. $\lim_{x \to 5} x^2 = 25$.

Proof. It's easy to show $\lim_{x\to 5} x = 5$ (and we'll do this next time.) Then, using part 2 of the limit theorem with $f(x) = g(x) = x$, we get

$$\lim_{x\to 5} x^2 = \lim_{x\to 5}(x \cdot x) = \left(\lim_{x\to 5} x\right)\left(\lim_{x\to 5} x\right) = 5 \cdot 5 = 25.$$

□

Notice what quick and clean work the limit theorem makes of this problem compared to our previous solution.

Week 2, Friday: Limit theorems; continuity; variations on the definition of a limit.

Last time we saw that even proving something intuitively obvious such as $\lim_{x\to 5} x^2 = 25$ using and ε- δ argument can be quite difficult. Imagine trying to prove that $\lim_{x\to 1} x^5 - x^3 + 2x^2 + 4 = 6$, also intuitively obvious, using an ε-δ argument? It might seem impossible. Today, we'll learn how to handle limits for all polynomials (and quotients of polynomials).

The strategy hinges on the following very general limit theorem:

Limit Theorem. Suppose that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then

1. $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.

2. $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$.

3. If $\lim_{x\to c} g(x) \neq 0$, then
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

In words, this theorem says that if we know the limits of f and g, then we can easily find the limit of any function we can construct from f and g using the operations of addition, multiplication, and division. For instance, combined with the following proposition, we can easily find the limit of any quotient of polynomials.

Proposition. Let c and k be real numbers. Then

1. $\lim_{x\to c} k = k$.

2. $\lim_{x\to c} x = c$.

Proof. For part 1, let $f(x) = k$, the constant function. We are trying to show $\lim_{x\to c} f(x) = k$. Given $\varepsilon > 0$, let $\delta = 1$ (in fact, this argument will work for any choice of positive δ). Then $0 < |x - c| < \delta$ implies

$$|f(x) - k| = |k - k| = 0 < \varepsilon.$$

(This proof my be a little confusing because the condition we need to be satisfied, $|f(x) - k|$ is trivially satisfied since f is such a trivial function.)

For part 2, the function in question is $g(x) = x$. Given $\varepsilon > 0$, let $\delta = \varepsilon$ and suppose $0 < |x - c| < \delta = \varepsilon$. Then

$$|g(x) - c| = |x - c| < \delta = \varepsilon,$$

as required. (Again, pretty trivial since g is so simple.)

To see how useful the above results are, let's go back to the function $f(x) = x^2$.

Example 1. $\lim_{x\to 5} x^2 = 25$.

Proof. By the above Proposition, we know $\lim_{x\to 5} x = 5$. Using part 2 of the limit theorem with $f(x) = g(x) = x$, we get

$$\lim_{x\to 5} x^2 = \lim_{x\to 5}(x \cdot x) = \left(\lim_{x\to 5} x\right)\left(\lim_{x\to 5} x\right) = 5 \cdot 5 = 25.$$

□

Much easier! Here is a useful result:

Claim ⋆. If $\lim_{x\to c} f(x)$ exists and $k \in \mathbb{R}$, then

$$\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x).$$

Proof. Combine part 2 of the limit theorem with the proposition:

$$\lim_{x\to c} kf(x) = \lim_{x\to c} k \ \lim_{x\to c} f(x) = k \ \lim_{x\to c} f(x).$$

Example 2. $\lim_{x\to 1} \frac{3x^2-5}{x^3-2x+3} = -1$.

Proof. Apply the limit theorem

$$
\begin{aligned}
\lim_{x\to 1}\frac{3x^2-5}{x^3-2x+3} &= \frac{\lim_{x\to 1}(3x^2-5)}{\lim_{x\to 1}(x^3-2x+3)} && \text{(part 3)}\\
&= \frac{\lim_{x\to 1}(3x^2-5)}{\lim_{x\to 1}(x^3-2x+3)} && \text{(part 1)}\\
&= \frac{\lim_{x\to 1}3x^2+\lim_{x\to 1}(-5)}{\lim_{x\to 1}x^3+\lim_{x\to 1}-2x+\lim_{x\to 1}3)} && \text{(part 1)}\\
&= \frac{3(\lim_{x\to 1}x^2)-5}{\lim_{x\to 1}x^3-2\lim_{x\to 1}x+\lim_{x\to 1}3} && (\star \text{ and prop.})\\
&= \frac{3(\lim_{x\to 1}x\lim_{x\to 1}x)-5}{\lim_{x\to 1}x\lim_{x\to 1}x\lim_{x\to 1}x-2\lim_{x\to 1}x+\lim_{x\to 1}3} && \text{(part 2)}\\
&= \frac{3(1\cdot 1)-5}{1\cdot 1\cdot 1-2\cdot 1+3} && \text{(prop.)}\\
&= \frac{-2}{2}\\
&= -1.
\end{aligned}
$$

□

Image how a straight ε-δ proof of this claim would look!

Where did εs and δ go in the proofs in Examples 1 and 2? The answer is that they are hidden inside the proof of our limit theorem. A complete proof of the limit theorem might appear in math 112. To give an idea, though, and to introduce the important "$\varepsilon/2$-trick", we'll prove part 1 of the theorem.

The complete proof of part (1) of the Limit Theorem will appear in the next lecture. For now, we'll consider the motivation for the proof. We are given that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Say $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. We must show that

$$\lim_{x\to c}(f(x)+g(x)) = L+M.$$

Given $\varepsilon > 0$, we will want

$$|(f(x)+g(x))-(L+M)| < \varepsilon,$$

when x is close to c. Since $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, we get to suppose that we can make both $|f(x)-L|$ and $|g(x)-M|$ small when x is close to c. However, note that

$$|(f(x)+g(x))-(L+M)| = |(f(x)-L)+(g(x)-M)| \qquad (\star)$$

If $f(x) - L$ and $g(x) - M$ are small, then the above equality gives us hope that we can make $|(f(x) + g(x)) - (L + M)|$ small, too. Looking ahead, if we want $|(f(x) + g(x)) - (L + M)| < \varepsilon$, in light of $(\star)$ it seems reasonable reasonable to require $|f(x) - M|$ and $|g(x) - M|$ to both be less that $\varepsilon/2$. We'll put these ideas together in a rigorous proof next time.

Week 3, Monday: Limit theorem proof; cancellation trick; rationalization trick.

Last time, we introduced the Limit Theorem, which allows us to give "high level" limit proofs, i.e., limit proofs that don't go all the way back to the definition, using εs and δs. The strategy for evaluating limits using the limit theorem is:

1. First compute the limits of some simple functions by hand. In our case, we showed how to evaluate the limit of a constant function and of the function $f(x) = x$.

2. Next, the limit theorem shows us how to compute limits for any function that we can build out of these simple functions using addition, multiplication, and division. Given constant function and $f(x) = x$ as our building blocks, for example, we can create all rational functions, i.e., all functions of the form $g(x)/h(x)$ where g and h are polynomials.

We now give a proof of part 1 of the Limit Theorem: if $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then

$$\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x).$$

We gave the motivation for the proof in the last lecture (and you might want to review that motivation before continuing). The proof will include the important "$\varepsilon/2$-trick" and depends on the following workhorse of analysis:

Triangle inequality. Let x and y be real numbers. Then

$$|x + y| \leq |x| + |y|.$$

PROOF. Math 112. $\square$.

For example,

$$3 = |-3 + 6| \leq |-3| + |6| = 3 + 6 = 9.$$

Proof of part 1 of the limit theorem. Suppose $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. We must show that

$$\lim_{x\to c}(f(x) + g(x)) = L + M.$$

Let $\varepsilon > 0$. We need to show that there is a $\delta > 0$ such that $0 < |x - c| < \delta$ implies

$$|(f(x) + g(x)) - (L + M)| < \varepsilon.$$

Since $\lim_{x\to c} f(x) = L$, we know that given any $\varepsilon' > 0$, there is a $\delta' > 0$ such that $0 < |x - c| < \delta'$ implies $|f(x) - L| < \varepsilon'$. In particular, there is such a δ' in the case where $\varepsilon' = \varepsilon/2$ (where ε is the number we need to beat, fixed above). To summarize: there exists $\delta' > 0$ such that $0 < |x - c| < \delta'$ implies

$$|f(x) - L| < \frac{\varepsilon}{2}.$$

By the same argument, since $\lim_{x\to c} g(x) = M$, there exists a $\delta'' > 0$ such that $0 < |x - c| < \delta''$ implies

$$|g(x) - M| < \frac{\varepsilon}{2}.$$

Recall that once we have one δ that works in the definition of the limit, then we can replace it by any smaller (positive) δ. So define $\delta = \min\{\delta', \delta''\}$, the minimum of δ' and δ''. It follows that if $0 < |x - c| < \delta$ we have both inequalities

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{and} \quad |g(x) - M| < \frac{\varepsilon}{2},$$

simultaneously. Using the triangle inequality, we then see that $0 < |x - c| < \delta$ implies

$$\begin{aligned} |(f(x) + g(x)) - (L - M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

as required. □

We will now introduce two important techniques for evaluating limits: the cancellation and the rationalization tricks.

Cancellation trick. Last time, we saw a typical example of using the limit theorem to evaluate the limit of a rational function (quotient of polynomials). Here, we'll give a similar example but where the limit theorem seems to break down.

$$\begin{aligned}
\lim_{x\to 2}\frac{x^2+x-6}{x-2} &= \frac{\lim_{x\to 2}(x^2+x-6)}{\lim_{x\to 2}(x-2)} \\
&= \frac{\lim_{x\to 2}x^2+\lim_{x\to 2}x+\lim_{x\to 2}(-6)}{\lim_{x\to 2}x+\lim_{x\to 2}(-2)} \\
&= \frac{\lim_{x\to 2}x\lim_{x\to 2}x+\lim_{x\to 2}x+\lim_{x\to 2}(-6)}{\lim_{x\to 2}x+\lim_{x\to 2}(-2)} \\
&= \frac{2\cdot 2+2-6}{2-2} \\
&= \frac{0}{0}.
\end{aligned}$$

What's the problem? Then answer is that part 3 of the limit theorem says that

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c}f(x)}{\lim_{x\to c}g(x)}$$

provided $\lim_{x\to c}g(x)\neq 0$. So the very first step of the argument above is not allowed since $\lim_{x\to 2}(x-2)=0$. Worse, it turns out that it is limits like this one that are most important in the study of calculus. Recall that in computing instantaneous speed for a distance function $f(t)$ at time $t=a$, we first compute the average speed over a time interval h:

$$\text{average\_speed}(h) = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

then take the limit as $h\to 0$. However, trying to apply the limit theorem, a similar problem arises since $\lim_{h\to 0}h=0$ in the denominator. So we can't directly use the limit theorem.

We now correctly calculate the limit in the example to show a typical way forward in this situation:

$$\begin{aligned}
\lim_{x\to 2}\frac{x^2+x-6}{x-2} &= \lim_{x\to 2}\frac{(x-2)(x+3)}{x-2} \\
&= \lim_{x\to 2}(x+3) \\
&= \lim_{x\to 2}x+\lim_{x\to 2}3 \\
&= 2+3 \\
&= 5.
\end{aligned}$$

In the second step, we replace the function

$$\frac{(x-2)(x+3)}{x-2}$$

with the function

$$x+3.$$

The reason this is OK is that these two functions are equal for all x except $x = 2$. When $x = 2$, the first function is undefined while the second function is $2 + 3 = 5$. *However*, when calculating the limit, recall that we only worry about x such that $0 < |x - c| < \delta$. Since $0 < |x - c|$, this means we never consider the case where $x = c$, or in our case, $x = 2$. So for the purpose of our limit, we are allowed to substitute $x + 3$ for $(x - 2)(x - 3)/(x - 3)$.

Rationalization trick The cancellation trick, just illustrated above, comes up fairly often in calculus. Here is another, somewhat less common, trick. It relies on the fact that $(a - b)(a + b) = a^2 - b^2$ and that multiplying the top and the bottom of a fraction by the same thing does not change the value of the fraction (since it amounts to multiplying by 1). We will need to use the fact that $\lim_{x\to 6} \sqrt{x+3} + 3 = 6$, which we'll just assume for now (it should seem reasonable) and prove later.
Consider the limit

$$\lim_{x\to 6} \frac{\sqrt{x+3}-3}{x-6}.$$

The first thing you should consider is a straightforward application of our limit theorem. That amount plugging in 6 for x. Unfortunately, $\lim_{x\to 6}(x - 6) = 0$ in the denominator. However, there is hope for cancellation since $\lim_{x\to 6}(\sqrt{x+3} - 3) = 0$, too. Here is how the computation goes:

$$\begin{aligned}
\lim_{x \to 6} \frac{\sqrt{x+3}-3}{x-6} &= \lim_{x \to 6} \frac{\sqrt{x+3}-3}{x-6} \cdot \frac{\sqrt{x+3}+3}{\sqrt{x+3}+3} \\
&= \lim_{x \to 6} \frac{(\sqrt{x+3}-3)(\sqrt{x+3}+3)}{(x-6)(\sqrt{x+3}+3)} \\
&= \lim_{x \to 6} \frac{(x+3)-9}{(x-6)(\sqrt{x+3}+3)} \\
&= \lim_{x \to 6} \frac{x-6}{(x-6)(\sqrt{x+3}+3)} \\
&= \lim_{x \to 6} \frac{1}{\sqrt{x+3}+3} \\
&= \frac{1}{6}.
\end{aligned}$$

Week 3, Wednesday: Continuity, compositions of continuous functions.

Last time, we used the rationalization trick to show that

$$\lim_{x\to 6} \frac{\sqrt{x+3}-3}{x-6} = \frac{1}{6}.$$

At one point in the calculation we needed to use the fact that

$$\lim_{x\to 6}(\sqrt{x+3}+3) = 6.$$

Our immediate goal is prove that now, using it as an excuse to introduce two important ideas: (I) continuity and (II) limits of compositions of functions.

I. Continuity. The first instinct in evaluating $\lim_{x\to c} f(x)$ is to just stick in c for x. In other words, there is a natural tendency to say $\lim_{x\to c} f(x) = f(c)$. For most of the functions you've dealt with in the past, this way of thinking is valid, and if these were the only kind of functions, there would be no need for the concept of a limit. These best-behaved functions are called *continuous* functions.

Definition. The function f is *continuous at a point* $c \in \mathbb{R}$ if

$$\lim_{x\to c} f(x) = f(c).$$

If f is continuous at every point, we simply say f is a *continuous* function.

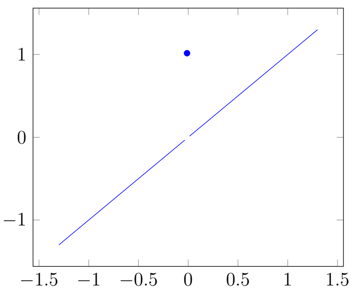
Since $\lim_{x\to c} x = c$, another way to say f is continuous is to say $\lim_{x\to c} f(x) = f(\lim_{x\to cx})$ for all c. In other words, you can bring the limit side inside, or "f commutes with limits".

Examples of continuous functions include any polynomial or quotient of polynomials (if you don't divide by zero). This is easily proved with our Limit Theorem. Other continuous functions are $\sqrt{x}$, e^x, $\cos(x)$, $\sin(x)$, to name a few. The following function is not continuous:

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

To check it's not continuous, first note that if $x \neq 0$, we have $f(x) = x$, thus, taking the limit as $x \to 0$, we have

$$\lim_{x \to x} f(x) = \lim_{x \to 0} x = 0 \neq f(0) = 1.$$



Graph of the function $f(x)$ defined above.

We are allowed the first equal in the above calculation because we never consider the case $x = 0$ when evaluating the limit as $x \to 0$. (Recall that in the definition of the limit, we consider x such that $0 < |x - c| < \delta$. So we don't care what happens when $x = c$.)

II. Composition of functions. Recall that if $f(x)$ and $g(x)$ are functions, then the *composition* of f and g is the function:

$$(f \circ g)(x) := f(g(x)).$$

(Note: the symbol ":=" means "the thing on the left is defined by the thing on the right".) So to compose f and g, you first evalute g, then stick the result into f.

Examples. (i) If $f(x) = x^3$ and $g(x) = x + 2$, then

$$(f \circ g)(x) = f(g(x)) = f(x + 2) = (x + 2)^3.$$

(ii) If $f(x) = \sqrt{x}$ and $g(x) = x + 3$, then

$$(f \circ g)(x) = f(g(x)) = f(x + 3) = \sqrt{x + 3}.$$

Composition theorem. If f and g are continuous functions, then

$$\lim_{x \to c} (f \circ g)(x) = f(\lim_{x \to c} g(x)).$$

Proof. Math 112. $\square$

III. $\lim_{x\to 6}\sqrt{x+3} = 6$**.**

Proposition. The function $\sqrt{x}$ is continuous, i.e.,

$$\lim_{x\to c}\sqrt{x} = \sqrt{c},$$

for all $c \geq 0$.

Proof. Let $f(x) = \sqrt{x}$. Our task is to show $\lim_{x\to c} f(x) = \sqrt{c}$. For homework, you have already shown that $\lim_{x\to 0} f(x) = f(0) = 0$, which shows that $f(x)$ is continuous at $x = 0$. So we may assume that $c > 0$.

Given $\varepsilon > 0$, let $\delta = \sqrt{c}\,\varepsilon$, and suppose

$$0 < |x - c| < \delta = \sqrt{c}\,\varepsilon.$$

Then using the relation $a^2 - b^2 = (a+b)(a-b)$.

$$\begin{aligned}\sqrt{c}\,\varepsilon &> |x - c| \\ &= |\sqrt{x} + \sqrt{c}||\sqrt{x} - \sqrt{c}|\end{aligned}$$

Since $\sqrt{x} \geq 0$, we have $|\sqrt{x}+\sqrt{c}| > |\sqrt{c}| = \sqrt{c}$. Therefore, continuing our calculation,

$$\begin{aligned}\sqrt{c}\,\varepsilon &> |x - c| \\ &= |\sqrt{x} + \sqrt{c}||\sqrt{x} - \sqrt{c}| \\ &= \sqrt{c}\,|\sqrt{x} - \sqrt{c}|.\end{aligned}$$

Canceling $\sqrt{c}$ yields

$$\varepsilon > |\sqrt{x} - \sqrt{c}| = |f(x) - \sqrt{c}|,$$

as desired. □.

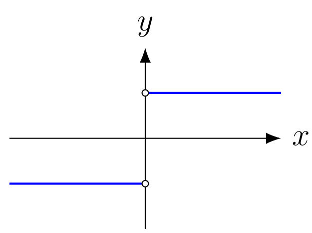
Week 3, Friday: Variations on the definition of the limit. The intermediate value theorem.

Today, we'll talk about two things: variations on the definition of the limit, and the intermediate value theorem (IVT).

I. VARIATIONS ON THE DEFINITION OF THE LIMIT.

Right- and left-hand limits. We'll start with an example. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$



Graph of $f(x)$.

As you might guess from the picture, $\lim_{x\to 0} f(x)$ does not exists (for instance, you can't beat $\varepsilon = 2$ with any δ). However, only think of the right-hand side of this graph, the limit looks like it should be 1, and if you only think of the left-hand side, the limit looks like it should be -1. The following definitions make this intuition precise.

Definition. *Right-hand limit:* $\lim_{x\to c^+} f(x) = L$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < x - c < \delta$ implies $|f(x) - L| < \varepsilon$.

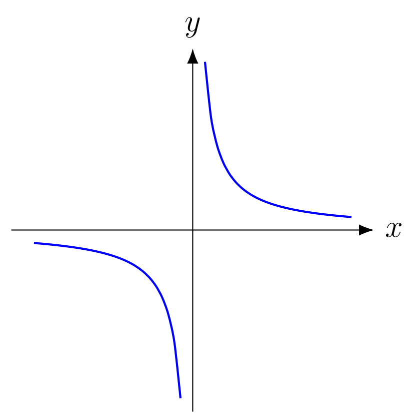
Left-hand limit $\lim_{x\to c^-} f(x) = L$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < c - x < \delta$ implies $|f(x) - L| < \varepsilon$.

Remark. Not that these definitions look very similar to the ordinary definition of the limit. The only difference is that the the condition $0 < |x - c| < \delta$ is replaced by either $0 < x - c < \delta$ or $0 < c - x < \delta$. Note that $0 < x - c$ just means that the x we will consider are strictly greater than c, i.e., to the right of c. Similarly, $0 < c - x$ means that x is to the left of c. In the ordinary definition of the limit, the x-values can be to the right or to the left of c. Hence, the absolute values: $0 < |x - c| < \delta$.

Example. For the function f given at the beginning of this lecture, we have

$$\lim_{x \to 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -1.$$

Limits at ∞. Below, we graph the function $g(x) = 1/x$:



Graph of $g(x) = 1/x$.

As x gets very large, in the positive or negative direction, the function $g(x) = 1/x$ gets very close to 0. So it is tempting to take the limit as x goes to ∞ or to $-\infty$. The trouble with trying to use our definition of the limit to make sense of the statement $\lim_{x \to \infty} = 0$ is that the condition $0 < |x-c| < \delta$ would become $0 < |x-\infty| < \delta$, which does not make sense. What would $x - \infty$ mean? So to make our intuition precise requires a different definition, which we give below:

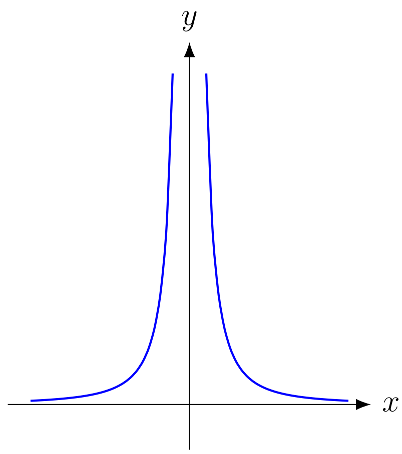
Definition. We say $\lim_{x \to \infty} f(x) = L$ if for all $\varepsilon > 0$, there exists N such that if $x > N$, then $|f(x) - L| < \varepsilon$. We say $\lim_{x \to -\infty} f(x) = L$ if for all $\varepsilon > 0$, there exists N such that if $x < N$, then $|f(x) - L| < \varepsilon$.

Remark. The condition $x > N$ should be thought of as saying "if x is really larger". Similarly, the condition $x < N$ should be though of as saying "if x is really negative".

Example. With these definitions, we have

$$\lim_{x\to\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0.$$

Infinite limits. Now consider the function $h(x) = 1/x^2$, whose graph appears below:



Graph of $g(x) = 1/x$.

What happens as $x \to 0$? This situation is somewhat similar to the previous one. Here is the definition we need:

Definition. We say $\lim_{x\to c} f(x) = \infty$ if for all N, there exists $\delta > 0$ such that $0 < |x - c| < \infty$ implies $f(x) > N$. We say $\lim_{x\to c} f(x) = -\infty$ if for all N, there exists $\delta > 0$ such that $0 < |x - c| < \infty$ implies $f(x) < N$.

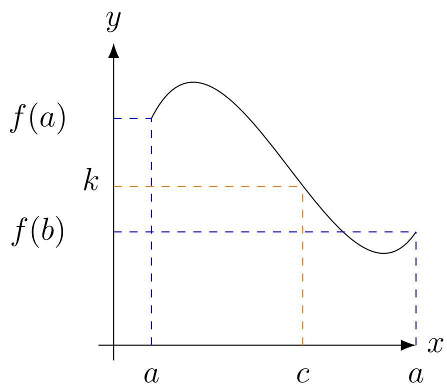
Remark. The condition $f(x) > N$ should be thought of as saying "$f(x)$ is really large. Similarly, $f(x) < N$ should be thought of a saying that "$f(x)$ is really negative.

Example. With these definitions, we have

$$\lim_{x\to 0} \frac{1}{x^2} = 0.$$

Intermediate Value Theorem (IVT). If f is a continuous function on a closed interval $[a, b]$, and k is a number between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = k$.

Proof. Math 112. $\square$



Graph of $f(x)$.

Corollary. Suppose f is continuous on $[a, b]$. If $f(a)$ and $f(b)$ have opposite signs, then there exists $c \in [a, b]$ such that $f(c) = k$.

Proof. If $f(a)$ and $f(b)$ have opposite signs, then $k = 0$ is between $f(a)$ and $f(b)$. Apply the IVT. $\square$

Example. Consider the polynomial $f(x) = x^5 + x + 1$. Then f is continuous since it's a polynomial. Since $f(-1) = -1$ and $f(0) = 1$, by the IVT, we know there is a $c \in [-1, 0]$ such that $f(c) = 0$. To find a more precise locate for a point where f is 0, compute $f(-0.5)$. We find $f(-0.5) = 0.46875$, which is positive. We know that $f(-1)$ is negative. Thus, by the IVT, there is a $c \in [-1, -0.5]$ such that $f(c) = 0$. Now evaluate f at the midpoint of $[-1, -0.5]$ and check out its sign. This helps narrow the locate of a zero of f even further. You can repeat this process, dividing an interval in half at each step to quickly approximate a zero of the function f.

Week 4, Monday: Definition of the derivative.

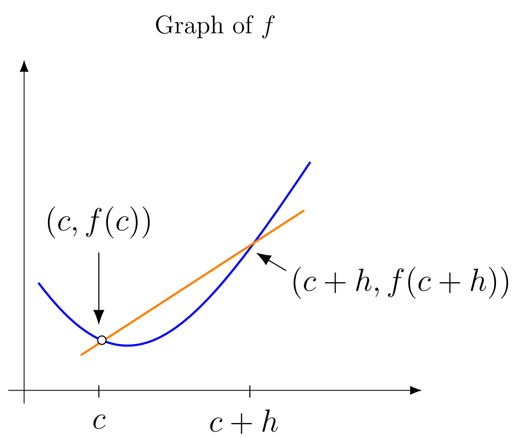
THE DERIVATIVE

Definition. The *derivative* of the function f at the point c is

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h},$$

provided the limit exists. If it does, then f is *differentiable at* c. If f is differentiable at all point where it is defined, then we say that f is *differentiable*. The derivative at $x = c$ is also known as the *slope of* f *at* c or the *instantaneous rate of change of* f *at* c.

The picture below illustrated the geometric meaning of the derivative:



The slope of the orange *secant* line is

$$\frac{f(c+h) - f(c)}{h}.$$

This is the *average rate of change* of f from $x = c$ to $x = c + h$. If f is thought of as giving the distance of a particle along the y-axis, then the quotient above is the *average speed* between times $x = c$ and $x = c + h$. To picture the derivative, imagine what happens to the above picture as h becomes smaller and smaller. The orange secant line will get less and less steep (for the function pictured). In the limit, we will get a line, called the *tangent to* f *at* c that touches the graph just at the point $(c, f(c))$. Its slope is the derivative.

Estimating $f'(x)$. Suppose we have the following table of values for a function f:

x	0.99	1.00	1.01	1.10
$f(x)$	1.8	2	2.03	2.1

To estimate $f'(1)$, we compute the average rate of change over near $x = 1$, i.e., use the approximation

$$f'(1) \approx \frac{f(1+h) - f(h)}{h}$$

for h small. For example, with $h = 0.1$,

$$f'(1) \approx \frac{f(1+0.1) - f(1)}{0.1} = \frac{2.1 - 2}{0.1} = 1,$$

or better (probably), we could take $h = 0.01$ to get

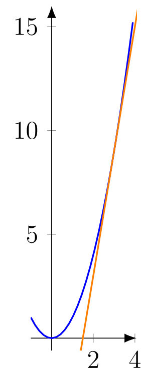
$$f'(1) \approx \frac{f(1+0.01) - f(1)}{0.01} = \frac{2.03 - 2}{0.01} = 3.$$

Calculating the derivative using the definition.

1. Find the derivative of $f(x) = x^2$ at $x = 3$.

   SOLUTION:

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h} \\ &= \lim_{h\to 0} \frac{(9 + 6h + h^2) - 9}{h} \\ &= \lim_{h\to 0} \frac{6h + h^2}{h} \\ &= \lim_{h\to 0} (6 + h) \\ &= 6. \end{aligned}$$



Graph of $f(x) = x^2$ and its tangent line at $x = 3$.

2. Find the derivative of $f(x) = x^2$ at an arbitrary point x.

   SOLUTION:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{(x^2 + 2hx + h^2) - x^2}{h} \\ &= \lim_{h\to 0} \frac{2hx + h^2}{h} \\ &= \lim_{h\to 0} (2x + h) \\ &= 2x. \end{aligned}$$

   Thus, $f'(x) = 2x$. Notation: we may also write $(x^2)' = 2x$. Letting $x = 3$ gives our previous result: $f'(3) = 2 \cdot 3 = 6$. In general, $f'(x)$ gives the slope of f. The graph of f is a parabola, and it makes sense that its slope increases as x increases. Our formula shows that wherever $x < 0$, the slope is negative, and wherever $x > 0$, the slope is positive. At $x = 0$ the slope is $f'(0) = 0$, which also makes sense: the tangent line there is horizontal.

3. Find the derivative of $f(x) = 5x + 7$ at an arbitrary point x.

SOLUTION:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{(5(x+h)+7)-(5x+7)}{h} \\ &= \lim_{h\to 0}\frac{5h}{h} \\ &= \lim_{h\to 0} 5 \\ &= 5. \end{aligned}$$

Thus, $f'(x) = 5$ at every point x. This says the slope of the function is 5 as every point, which makes sense since the graph of f is a line with slope 5.

4. Find the derivative of $f(x) = x^n$ at an arbitrary point x.

SOLUTION:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{(x+h)^n-x^n}{h} \\ &= \lim_{h\to 0}\frac{(x^n+nx^{n-1}h+\frac{n(n-1)}{2}x^{n-2}h^2+\cdots+nxh^{n-1}+h^n)-x^n}{h} \\ &= \lim_{h\to 0}\frac{nx^{n-1}h+\frac{n(n-1)}{2}x^{n-2}h^2+\cdots+nxh^{n-1}+h^n}{h} \\ &= \lim_{h\to 0}(nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}h+\cdots+nxh^{n-2}+h^{n-1}) \\ &= nx^{n-1}. \end{aligned}$$

For example the derivative of the function $f(x) = x^5$ at an arbitrary point x is $5x^4$. For instance, the slope of $f(x) = x^5$ at the point 2 is $5 \cdots (2^4) = 80$. The case $n = 2$ recaptures our previous result for $f(x) = x^2$, i.e., it's derivative is $2x$.

Equation of the tangent line. The *tangent line* for $f(x)$ at a point $x = c$ is the line with slope $f'(c)$ and passing through the point $(c, f(c))$. It has the equation

$$\frac{y - f(c)}{x - c} = f'(c),$$

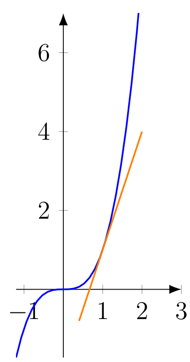
or, solving for y:

$$y = f(c) + f'(c)(x - c).$$

Example. To find the tangent line to $f(x) = x^3$ at the point $x = 1$, we first calculate the derivative of $f(x) = x^3$ at $x = 1$ using the formula we computed above: $f'(x) = 3x^2$, hence, the slope is $f'(1) = 3$. The tangent line has equation

$$y = f(1) + f'(2)(x - 1) = 1 + 3(x - 1) = 3x - 2.$$

In sum, the tangent line at $x = 1$ has equation $y = 3x - 2$.



Graph of $f(x) = x^3$ and its tangent line at $x = 1$.

Week 4, Wednesday: Instantaneous change, tangent lines; first properties of derivatives.

Warm-up and review. Suppose the position of a particle along the y-axis is given by $f(x) = \sqrt{x}$.

1. What is the average speed of the particle between times $x = 1$ and $x = 4$?

   SOLUTION:
$$\text{average speed} = \frac{f(4) - f(1)}{4 - 1} = \frac{\sqrt{4} - \sqrt{1}}{3} = \frac{1}{3}.$$

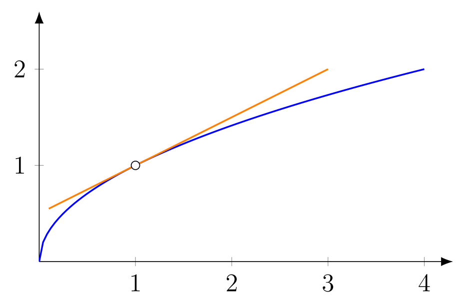
2. What is the instantaneous speed of the particle at time $x = 1$?

   SOLUTION:
$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} \\ &= \lim_{h \to 0} \frac{(1+h) - 1}{h(\sqrt{1+h} + 1)} \\ &= \lim_{h \to 0} \frac{h}{h(\sqrt{1+h} + 1)} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{1+h} + 1} \\ &= \lim_{h \to 0} \frac{1}{2}. \end{aligned}$$

3. Find the equation for the tangent line to f at $x = 1$.

   SOLUTION: We have just calculated the slop of that line: $f'(1) = 1/2$. Therefore, the line has equation

$$\frac{y - f(1)}{x - 1} = f'(1) = \frac{1}{2} \Rightarrow y = 1 + \frac{1}{2}(x-1) \Rightarrow y = 1 + \frac{1}{2}x - \frac{1}{2}.$$



Graph of $f(x) = \sqrt{x}$ and its tangent line at $x = 1$.

4. What is the instantaneous speed at an arbitrary time x.

   SOLUTION:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h \to 0} \frac{1}{2\sqrt{x}}.
\end{aligned}$$

5. What are $\lim_{x\to 0^+} f'(x)$ and $\lim_{x\to\infty} f'(x)$, and what does this mean geometrically?

SOLUTION: We won't prove this, but since $f'(x) = 1/2\sqrt{x}$, it turns out that $\lim_{x\to 0^+} f'(x) = \infty$ and $\lim_{x\to\infty} f'(x) = 0$. Note from the graph of $f(x)$, drawn above, that the slope of the graph approaches ∞ as $x \to 0^+$ and approaches 0 as $x \to \infty$.

First properties of derivatives.

Recall our limit theorem which told us how to build up complicated limits from the limits of simple functions. There is a similar result for derivatives, but it has a very interesting twist when it comes to products of functions.

Theorem. Suppose f and g are differentiable functions at a point x.

1. The derivative of a constant function is 0: Let $c \in \mathbb{R}$, and let $h(x) = c$. or written in different notation, $(c)' = 0$. (Note that the graph of a constant function is a straight line with slope 0. So this makes sense.)

2. Let $k(x) = x$. Then $k'(x) = 1$, i.e., $(x)' = 1$. (Note that the graph of k is a line with slope 1.)

3. $(f(x)+g(x))' = f'(x)+g'(x)$: the derivative of a sum is the sum of the derivatives.

4. The *product rule or Leibniz rule.*
$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

5. The *quotient rule.*
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

We'll prove parts of this theorem later. They follow straight from the definition of the derivative by taking limits. For now, let's play with the theorem to see what it tells us. First, some notation to save time. Instead of writing $(f(x)g(x))' = f'(x)g(x) + f(x)g(x)$, we'll often write $(fg)' = f'g + fg'$, dropping the x. Similarly, we'll write
$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

One consequence of the theorem is that when we compute derivatives we can "pull constants out" in the following sense: Let c be a constant. Then by the product rule:
$$(cf)' = (c)'f + cf' = 0 \cdot f + cf' = cf'.$$

For example,

$$(5x)' = 5(x)' = 5 \cdot 1 = 5.$$

Here is another example of a consequence of the theorem:

$$(f - g)' = f' + (-g)' = f' + (-1 \cdot g)'.$$

Continuing, using the fact that we can pull constants out:

$$= f' - 1 \cdot (g)' = f' - g'.$$

So $(f - g)' = f' - g'$.

Finally, note that the function $f(x) = x^n$ where n is any number $n = 1, 2, \ldots$ can be written as a product of functions: $x^n = x \cdot x \cdots x$. So we can apply the product rule to find its derivative. For instance,

$$(x^2)' = (x \cdot x)' = (x')x + x(x)' = 1 \cdot x + x \cdot 1 = 2x.$$

Knowing that $(x^2)' = 2x$, we find

$$(x^3)' = (x^2 \cdot x) = (x^2)'x + x^2 \cdot (x)' = 2x \cdot x + x^2 \cdot 1 = 3x^2.$$

Knowing that $(x^3)' = 3x^2$, we find

$$(x^4)' = (x^3 \cdot x) = (x^3)'x + x^3 \cdot (x)' = 3x^2 \cdot x + x^3 \cdot 1 = 4x^3.$$

Continuing in this way we get

$$(x^n) = nx^{n-1}$$

for $n = 1, 2, \ldots$

Combining that result with the derivative theorem, we can now compute the derivative of any polynomial (or quotient of polynomials). For example:

$$\begin{aligned}
(3x^5 - 2x^2 - 7)' &= (3x^5)' + (-2x^2)' + (-7)' \\
&= 3(x^5)' - 2(x^2)' + 0 \\
&= 3(5x^4) - 2(2x) \\
&= 15x^4 - 4x.
\end{aligned}$$

Week 4, Friday: Proof of derivative theorem.

Our goal today is to prove the "derivative theorem" presented last time.
We'll need the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists, and we'll need our earlier limit theorem:
Limit Theorem. Suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. Then

1. $\lim_{x \to c}(f(x) + g(x)) = L + M$,
2. $\lim_{x \to c} f(x)g(x) = LM$,
3. if $M \neq 0$, then
$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Derivative theorem. Suppose f and g are differentiable functions at a point x. Then

1. $(f(x) + g(x))' = f'(x) + g'(x)$,
2. *product rule or Leibniz rule*:
$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$
3. *quotient rule*:
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Proof. Part 1:

$$
\begin{aligned}
(f(x)+g(x))' &= \lim_{h\to 0} \frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h} \\
&= \lim_{h\to 0} \frac{(f(x+h)-f(x))+(g(x+h)-g(x))}{h} \\
&= \lim_{h\to 0} \left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} \right) \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= f'(x)+g'(x).
\end{aligned}
$$

Notice how we used our earlier limit theorem to say that the limit of a sum is the sum of the limits.

Part 2. This one's a bit trickier—it involves subtracting and adding $f(x)g(x+h)$:

$$
\begin{aligned}
(f(x)g(x))' &= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left(\frac{(f(x+h)-f(x))g(x+h)}{h} + \frac{f(x)(g(x+h)-g(x))}{h} \right) \\
&= \lim_{h\to 0} \frac{(f(x+h)-f(x))g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)(g(x+h)-g(x))}{h} \\
&= \lim_{h\to 0} \frac{(f(x+h)-f(x))}{h} \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \lim_{h\to 0} \frac{(g(x+h)-g(x))}{h}.
\end{aligned}
$$

For the last two steps in the above calculation, we used parts 1 and 2 of the limit theorem. Continuing, now use the definition of the derivative:

$$(f(x)g(x))' = f'(x) \lim_{h\to 0} g(x+h) + \left(\lim_{h\to 0} f(x) \right) g'(x).$$

We have $\lim_{h\to 0} f(x) = f(x)$ since $f(x)$ can be thought of as a constant function of h. Therefore,

$$(f(x)g(x))' = f'(x) \left(\lim_{h\to 0} g(x+h) \right) + f(x)g'(x).$$

Finally, we use a fact the we may or may not prove later: differentiable functions are continuous, i.e., we can evaluate their limits by just plugging in the limit point. In particular, g is continous. Then $g(x+h)$ is a composition of continuous functions of h: $g(x+h) = g(k(h))$ where $k(h) = x+h$. Therefore, $\lim_{h\to 0} g(x+h) = g(x+0) = g(x)$. So we finally get

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

For Part 3, we first leave the following as an exercise for the reader:

$$\left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{g^2(x)}$$

where $g^2(x) = g(x) \cdot g(x)$. We combine this with the product rule of Part 1 to get

$$\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \left(f(x) \cdot \frac{1}{g(x)}\right)' \\
&= f'(x) \cdot \frac{1}{g(x)} + f(x)\left(\frac{1}{g(x)}\right)' \\
&= \frac{f'(x)}{g(x)} + f(x)\left(-\frac{g'(x)}{g^2(x)}\right) \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)},
\end{aligned}$$

where in the last step, we've just found a common denominator. □

Last time, we saw that this theorem makes calculating derivatives a lot easier than having to go back to the definition of the derivative every time. We saw that knowing $(c)' = 0$ for a constant $c \in \mathbb{R}$, and $(x)' = 1$, we can use the theorem to compute the derivative of any rational function. For instance, repeated use of the product rule allowed us the compute $(x^n)' = nx^{n-1}$ for $n = 1, 2, \ldots$ Using that fact and the derivative theorem, we can evaluate derivatives of all polynomials. For example,

$$\begin{aligned}
\left(3x^4 + x^2 - 4x + 2\right)' &= (3x^4)' + (x^2)' + (-4x)' + (2)' \qquad \text{(Part 1)} \\
&= 3(x^4)' + (x^2)' - 4(x)' + 0 \quad \text{(pulling out constants—see the last lecture)} \\
&= 3(4x^2) + (2x) - 4(1) \\
&= 12x^2 + 2x - 4.
\end{aligned}$$

With practice, you can perform all of these steps at once:

$$(6x^5 - 4x^3 + 12x^2 - 7x + 2)' = 30x^4 - 12x^2 + 24x - 7.$$

Now we apply the quotient rule to compute $(x^n)'$ for $n < 0$. To start,

$$(x^{-1})' = \left(\frac{1}{x}\right)' = -\frac{(x)'}{x^2} = -\frac{1}{x^2}.$$

Using this, we get

$$(x^{-2})' = \left(\frac{1}{x^2}\right)' = -\frac{(x^2)'}{(x^2)^2} = -\frac{2x}{x^4} = -\frac{2}{x^3}.$$

Continuing in this way, we get

$$\left(\frac{1}{x^n}\right)' = -\frac{n}{x^{n+1}}.$$

for $n = 1, 2, 3, \ldots$

Here is an example of a computation of the derivative of a typical rational function using the quotient rule:

$$\begin{aligned}
\left(\frac{x^2}{x^4+3x+2}\right)' &= \frac{(x^2)'(x^4+3x+2) - x^2(x^4+3x+2)'}{(x^4+3x+2)^2} \\
&= \frac{2x(x^4+3x+2) - x^2(4x^3+3)}{(x^4+3x+2)^2} \\
&= \frac{2x^5+6x^2+4x-4x^5-3x^2}{(x^4+3x+2)^2} \\
&= \frac{-2x^5+3x^2+4x}{(x^4+3x+2)^2}.
\end{aligned}$$

Week 5, Monday: Chain rule.

THE CHAIN RULE

[Note: please take a look at the essential derivatives handout at our class homepage.]

Today, we'll do two things: the chain rule, and a trigonometry review.

Sometimes a function can be constructed from simpler functions using addition, multiplication, and division. In that situation the sum, product, and quotient rules for derivatives (our derivative theorem) gives us a means of computing the derivative of the function in terms of the derivatives of its simpler constituent functions. Another way to build a function is by composing simpler functions. The chain rule tells you how the derivative behaves in this situation.

Theorem. Suppose f and g are differentiable functions. Then

$$(f(g(x)))' = f'(g(x))g'(x).$$

Proof. Math 202 (where it is proved for functions of several variables—the case of one-variable calculus being a special case). □

Example. Compute the derivative of $(3x^4 + 2x^3 + 5x + 2)^{25}$. There are two ways to do this. The first is to expand out the 25-th power of a polynomial, then take the derivative of this polynomial. That would be quite painful. The second way is to use the chain rule by recognizing the given function as the composition of two simpler functions: we have

$$(3x^4 + 2x^3 + 5x + 2)^{25} = f(g(x))$$

where

$$f(x) = x^{25} \quad \text{and} \quad g(x) = 3x^4 + 2x^3 + 5x + 2.$$

To apply the chain rule, first take the derivatives of f and g:

$$f'(x) = 25x^{24} \quad \text{and} \quad g'(x) = 12x^3 + 6x^2 + 5.$$

Next, use the chain rule (you will need to compose f' and g):

$$\begin{aligned}((3x^4+2x^3+5x+2)^{25})' &= (f(g(x)))' \\ &= f'(g(x))g'(x) \\ &= 25(3x^4+2x^3+5x+2)^{24}(12x^3+6x^2+5).\end{aligned}$$

Example. We have seen in class that $(\sqrt{x})' = 1/(2\sqrt{x})$. We can use this to find the derivative of $\sqrt{3x^2+2x+2}$:

$$(\sqrt{3x^2+2x+2})' = \frac{1}{2\sqrt{3x^2+2x+2}} \cdot (6x+2) = \frac{3x+1}{\sqrt{3x^2+2x+2}}.$$

Here, $f(x) = \sqrt{x}$ and $g(x) = 3x^2+2x+2$.

Example. Given that $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$, the chain rule gives us:

1. $$(\cos^3(x))' = 3\cos^2(x)(\cos(x))' = 3\cos^2(x)(-\sin(x)) = -3\cos^2(x)\sin(x).$$
   (Chain rule with $f(x) = x^3$, $g(x) = \cos(x)$.)

2. $$(\sin(\cos(x)))' = \cos(\cos(x))(\cos(x))' = -\cos(\cos(x))\sin(x).$$
   (Chain rule with $f(x) = \sin(x)$, $g(x) = \cos(x)$.)

For multiple compositions, we can apply the chain rule multiple times:

$$(f(g(h(x))))' = f'(g(h(x))g'(h(x))h'(x).$$

Examples.

$$\begin{aligned}(\sin((x^4+5x-2)^{10}))' &= \cos((x^4+5x-2)^{10})(10(x^4+5x-2)^9(4x^3+5)) \\ &= 10(x^4+5x-2)^9(4x^3+5)\cos((x^4+5x-2)^{10}).\end{aligned}$$

$$(\sin(\cos(x^3+5x)))' = \cos(\cos(x^3+5x))\sin(x^3+5x)(3x^2+5).$$

Example. You can, of course, combine the chain rule with the sum, product, and quotient rules. Here, we use the fact that $(\ln(x))' = 1/x$.

$$\begin{aligned}(x^2\sin(\ln(x)))' &= (x^2)'\sin(\ln(x)) + x^2(\sin(\ln(x)))' \\ &= 2x\sin(\ln(x)) + x^2\cos(\ln(x)(1/x) \\ &= 2x\sin(\ln(x)) + x\cos(\ln(x))\end{aligned}$$

Week 5, Wednesday: Trigonometry review. Implicit functions, and related rates.

Last time, we talked about the chain rule: if f and g are differentiable functions, then

$$(f(g(x))' = f'(g(x))g'(x).$$

Some examples of its application:

$$((3x^4 + 3x^2 + x + 1)^{25})' = 25(3x^4 + 3x^2 + x + 1)^{24}(12x^3 + 6x + 1)$$
$$(e^{\cos(x)})' = e^{\cos(x)}(-\sin(x)) = -\sin(x)e^{\cos(x)}.$$

It is common to see the chain rule expressed using a different notation which we will now describe. Suppose y is a function of x and x is a function of t. For instance, we could have

$$y = y(x) = x^3 \quad \text{and} \quad x = x(t) = t^2 + 4.$$

We can then express y as a function of t:

$$y = x^3 = (t^2 + 4)^3.$$

We can take the derivative of y, then, as a function of x or as a function of t. To make the distinction clear, we use the following notation:

$$\frac{dy}{dx} = 3x^2 \quad \text{and} \quad \frac{dy}{dt} = 3(t^2 + 4)^2(2t),$$

where we have used the chain rule in computing dy/dt. Also, note that $dx/dt = 2t$. In fact, using this notation, the chain rule can be expressed in this nice way:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

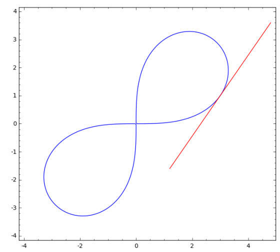
Explicitly, with our example,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (3x^2) \cdot (2t) = (3(t^2 + 4)^2)(2t).$$

Here is an application of these ideas. Suppose a function y is defined implicitly by the equation

$$3(x^2 + y^2)^2 = 100xy.$$

A picture of the points $(x, y) \in \mathbb{R}^2$ satisfying this equation appears below:



The picture show the tangent line to the curve at the point $(3, 1)$. Let's find the equation of this tangent line using the chai rule. The idea is to think of the equation as implicitly defining y as a function of x. Take the derivative of both sides of the equation with respect to x. For the right-hand side we get

$$\frac{d}{dx}(3(x^2 + y^2)^2) = 6(x^2 + y^2)\left(\frac{d}{dx}(x^2 + y^2)\right)$$

$$= 6(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right).$$

for the left-hand side, we apply the product rule and the chain rule to get

$$\frac{d}{dx}(100xy) = 100\left(\left(\frac{d}{dx}(x)\right)y + x\frac{dy}{dx}\right)$$

$$= 100\left(1\dot{y} + x\frac{dy}{dx}\right)$$

$$= 100\left(y + x\frac{dy}{dx}\right)$$

Setting the right-hand and left-hand sides equal, we get

$$6(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 100\left(y + x\frac{dy}{dx}\right).$$

We are interested in the point $(x, y) = (3, 1)$. Substituting $x = 3$ and $y = 1$ above:

$$60\left(6 + 2\frac{dy}{dx}\right) = 100\left(1 + 3\frac{dy}{dx}\right).$$

Solving for dy/dx now gives:

$$\frac{dy}{dx} = \frac{13}{9}.$$

So the slope of the tangent line is 13/9, and it passes through the point $(3, 1)$. So the tangent line has equation

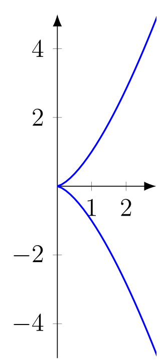
$$y = 1 + \frac{13}{9}(x - 3).$$

Week 5, Friday: Related rates.

Example. Here is one more example of computing the tangent line to an implicitly defined curve. Consider all of the point (x, y) that satisfy

$$y^2 = x^3.$$

A graph appears below:



The cuspidal cubic curve, $y^2 = x^3$.

The point $(1, 1)$ is on this curve since $1^3 = 1^2$. Let's compute an equation for the tangent line at that point. Taking the derivative of both sides of the defining equation with respect to x, we get

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^3) \Longrightarrow 2y\frac{dy}{dx} = 3x^2.$$

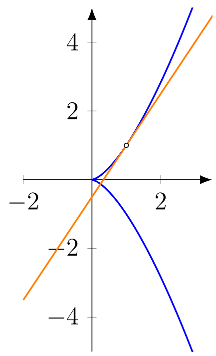
At the point $x = y = 1$, this becomes

$$2\frac{dy}{dx} = 3.$$

Hence, the slope at $(1, 1)$ is $3/2$. The line with slope $3/2$ and passing through the point $(1, 1,)$ has equation

$$y = 1 + \frac{3}{2}(x - 1).$$

Plotting this line with the curve gives the picture



The cuspidal cubic curve, $y^2 = x^3$.

In the case of this curve, it's easy to take the defining equation, $y^2 = x^3$, and solve for y. This yields

$$y = x^{3/2} \quad \text{or} \quad y = -x^{3/2}.$$

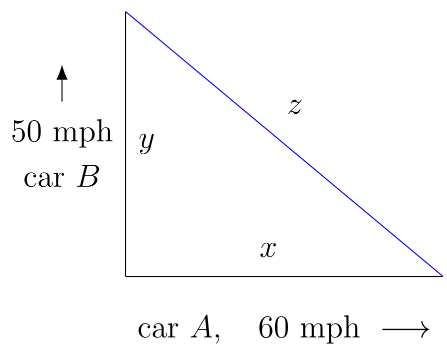
It's easy to see both branches of this function in the picture of the curve above. The part of the curve passing through $(1, 1)$ would be on the branch defined by $y = x^{3/2}$. We can now use ordinary methods to find the slope at $x = 1$:

$$(y)' = \frac{3}{2}x^{1/2}.$$

Evaluating at $x = 1$ gives $y'(1) = 3/2$. So the slope is $3/2$, as we found earlier.

Example 1. Cars A and B travel at right angles, starting at the same point. Car A travels at 60 mph and car B travels at 50 mph. How fast is the distance between the cars increasing at 2 hours?

SOLUTION: The **first step** in a related rates problem is to draw a picture of the situation, labeling all relevant quantities. In our case, the picture could be as below:



The **second step** is to write equations relating the relevant variables and stating what we are given in terms of the variables. In our case, we use the Pythagorean theorem and express the speed of the cars:

$$x^2 + y^2 = z^2, \quad \frac{dx}{dt} = 60, \quad \frac{dy}{dt} = 50.$$

Then restate our problem in terms of our variables. In our case, the problem becomes:

$$\text{Find } \frac{dz}{dt} \text{ when } t = 2.$$

The **third step** is to use the chain rule to differentiate the equation relating the variables. In our case, we differentiate $x^2 + y^2 = z^2$ with respect to t:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt},$$

or, dividing by 2,

$$x\frac{dx}{dt} + y\frac{dy}{dt} = \frac{dz}{dt}, \tag{14.1}$$

The **fourth step** (and final) is to subsitute into this equation all the quantities we know and solve for the quantity we are trying to determine. In our case, when $t = 2$, we have $x = 120$ and $y = 100$. From the equation $x^2 + y^2 = z^2$, we find

$$z = \sqrt{120^2 + 100^2} = 20\sqrt{61}.$$

Subsituting into equation (14.1), we get

$$120 \cdot 60 + 100 \cdot 50 = 20\sqrt{61}\,\frac{dz}{dt}.$$

Solving for dz/dt, we get the answer: the cars are moving apart at

$$\frac{dz}{dt} = \frac{1}{20\sqrt{61}}(7200 + 5000) = \frac{1}{10\sqrt{61}}(3600 + 2500) =\approx 78.1\,\text{mph}.$$

Question. As the time goes on, are the cars moving apart at the same rate at when they are after 2 hours? Faster? Slower?

SOLUTION: Let's figure out how quickly cars A and B are moving apart at an arbitrary time t. The set-up is the same as above. Move we want to find dz/dt at and arbitrary time t. Again, we use equation (14.1). After t hours, we have $x = 60t$ and $y = 50t$. Hence,

$$z = \sqrt{(60t)^2 + (50t)^2} = t\sqrt{60^2 + 50^2} = 10t\sqrt{61}.$$

Plugging into equation (14.1):

$$(60t) \cdot 60 + (50t) \cdot 50 = 10t\sqrt{61}\,\frac{dz}{dt}.$$

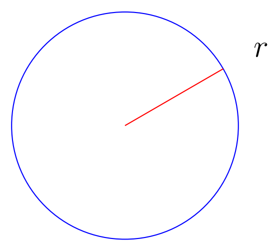
Canceling the t on both sides of this equation, we see that the value for dz/dt, in fact, does not depend on t. So the cars are forever moving apart at the same rate of about 78.1 mph.

Week 6, Monday: Related rates examples.

More related rates examples. (continuing from last class period)

Example 2. Suppose that oil is spilled on water and spreads in a circular pattern so that the radius is increasing at 2 ft/sec. How fast is the area of the spill increasing when the radius is 60 ft?

SOLUTION: **First step**—draw the relevant picture an assign names to the relevant variables:



Second step—write equations relating the relevant variables and stating what we are given in terms of the variables. We are asked to relate the radius to the area. So let A denote the area of the circle. The only relevant equation relating the variables is $A = \pi r^2$. In terms of these variables, we are being asked to find dA/dt when $r = 60$, and we are given that $dr/dt = 2$.

Third step—differentiate the equation relating the variables. Differentiating $A = \pi r^2$ with respect to time gives

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi r \cdot 2 = 4\pi r.$$

Fourth step—subsitute into this equation all the quantities we know and solve for the quantity we are trying to determine. When $r = 60$, we have $dA/dt = 4\pi \cdot 60 = 240\pi$ ft/sec.

Example 3. The ideal gas law (first formulated by Émile Clapeyron in 1834) states that

$$PV = nRT$$

where P is the pressure of the gas, V is its volume, n is the amount (in moles), R is a constant, and T is the absolute temperature. Suppose we have gas in a cylinder with a piston on one end. By moving the piston in and out, we can adjust the volume of the cylinder. Suppose we do this but maintain a constant temperature using some kind of apparatus attached to the outside of the cylinder. The ideal gas law that says that

$$PV = \text{ constant.}$$

So as we decrease the volume by pressing in on the piston, the pressure will then increase. Suppose we decrease the volume at a constant rate. Does the pressure increase at a constant rate?

SOLUTION: Suppose that $PV = c$ for some constant c. We are thinking of P and V as changing over time. So $P = P(t)$ and $V = V(t)$, i.e., both are functions of time. If the volume is decreasing at a constant rate, we can write

$$\frac{dV}{dt} = -k$$

for some $k > 0$. Take the derivative of $PV = c$ with respect to t using the product rule and the chain rule:

$$\begin{aligned} \frac{d}{dt}(PV) = \frac{d}{dt}(c) \quad &\Rightarrow \quad \frac{dP}{dt} \cdot V + P \cdot \frac{dV}{dt} = 0 \\ &\Rightarrow \quad \frac{dP}{dt} \cdot V - P \cdot k = 0. \end{aligned}$$

Solving for the rate of change of P over time:

$$\frac{dP}{dt} = \frac{P}{V} \cdot k.$$

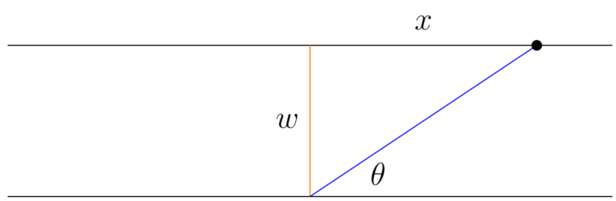
Now solve for P in $PV = c$ and substitute into the above equation to get

$$\frac{dP}{dt} = \frac{c}{V^2} \cdot k.$$

So as the volume gets smaller, the pressure increases not at a constant rate but according to an inverse-square law!

Example 4. Imagine a long hallway with paintings along the wall on one side. A slowly rotating surveillance camera is mounted on the opposite wall. How fast should its angle be changing in order for it to be scanning the opposite wall at a constant rate? Express your solution just in terms of the angle (which is what we would need to program the camera).

SOLUTION: Say the hall is some constant w feet wide. A picture of the situation is below:



We want the rate of change of x to be a constant. Say,

$$\frac{dx}{dt} = k.$$

The relation between θ and x is

$$\tan(\theta) = \frac{w}{x}. \tag{15.1}$$

Differentiating with respect to t gives, remembering that w is a constant and using the essential derivatives handout to find the derivative of the tangent function:

$$\begin{aligned}\frac{d}{dt}(\tan(\theta)) = \frac{d}{dt}\left(\frac{w}{x}\right) \quad &\Rightarrow \quad \frac{d}{dt}(\tan(\theta)) = w\frac{d}{dt}\left(\frac{1}{x}\right)\\ &\Rightarrow \quad \sec^2(\theta)\frac{d\theta}{dt} = -\frac{w}{x^2}\frac{dx}{dt}\\ &\Rightarrow \quad \sec^2(\theta)\frac{d\theta}{dt} = -\frac{kw}{x^2}.\end{aligned}$$

To express the solution solely in terms of θ, we can compute $1/x^2$ from equation (15.1):

$$\frac{1}{x^2} = \frac{\tan^2(\theta)}{w^2}.$$

Substitute this into the previous equation:

$$\sec^2(\theta)\frac{d\theta}{dt} = -\frac{kw}{x^2} = -kw \cdot \frac{\tan^2(\theta)}{w^2} = -k\frac{\tan^2(\theta)}{w}.$$

Solving for $d\theta/dt$:

$$\begin{aligned}\frac{d\theta}{dt} &= -k\frac{\tan^2(\theta)}{w} \cdot \frac{1}{\sec^2(\theta)}\\ &= -\frac{k}{w}\frac{\sin^2(\theta)}{\cos^2(\theta)} \cdot \cos^2(\theta)\\ &= -\frac{k}{w}\sin^2(\theta).\end{aligned}$$

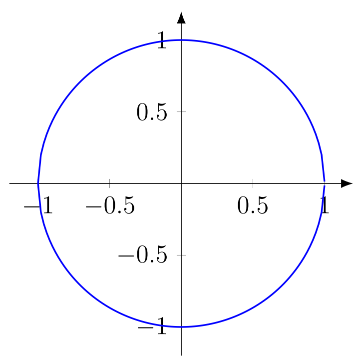
Does this answer make sense? The negative sign is OK: as x increases, θ is decreasing. Next, note that as θ goes to 0, we have $\sin(\theta)$ going to 0, too. This makes sense: if the hallway were infinitely long, the camera would need to move more and more slowly to keep the point x moving at a constant rate.

Week 6, Wednesday: Optimization.

Implicit differentiation. Here are a couple more examples of implicit differentiation.

Example 1. The unit circle centered at the origin has defining equation

$$x^2 + y^2 = 1.$$



The unit circle, $x^2 + y^2 = 1$.

This curve is not the graph of a function since there are multiple y-values for some of the x-values. (The top semicircle is the graph of $y = \sqrt{1 - x^2}$, and the bottom semicircle is the graph of $y = -\sqrt{1 - x^2}$.) We can compute the slope implicitly, though, by taking derivatives of both sides of the equation with respect to x:

$$\begin{aligned} x^2 + y^2 = 1 \quad &\Rightarrow \quad \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1) \\ &\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0 \\ &\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}. \end{aligned}$$

There is obviously trouble if $y = 0$. Note that the points on the circle where $y = 0$ are exactly the places where the tangent lines are vertical (having infinite slope). Note that $dy/dx = x/y = 0$ exactly where $x = 0$. These points occur at the top and bottom of the circle (the points $(0, \pm 1)$). For another example, consider the point $(\sqrt{2}/2, \sqrt{2}/2)$ on the unit circle. To find the slope at that point, plug into or equation for dy/dx:

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{\sqrt{2}/2}{\sqrt{2}/2} = -1.$$

Sure enough, the slope is -1, as expected.

Example 2. Consider the points (x, y) satisfying

$$\cos(xy) = \frac{1}{2}.$$

To compute the slope, use implicit differentiation (and the product rule):

$$\begin{aligned}
\cos(xy) = \frac{1}{2} \quad &\Rightarrow \quad \frac{d}{dx}\cos(xy) = \frac{d}{dx}\left(\frac{1}{2}\right) \\
&\Rightarrow \quad \frac{d}{dx}\cos(xy) = 0 \\
&\Rightarrow \quad -\sin(xy)\frac{d}{dx}(xy) = 0 \\
&\Rightarrow \quad -\sin(xy)\left(y + x\frac{dy}{dx}\right) = 0
\end{aligned}$$

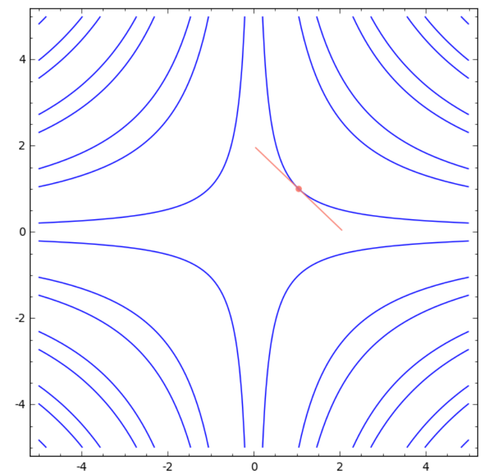
As long as $\sin(xy) \neq 0$ and $y \neq 0$, this implies

$$\frac{dy}{dx} = -\frac{y}{x}.$$

For example, let's pick a point satisfying $\cos(xy) = 1/2$, such as $x = \pi/3$ and $y = 1$. The slope of the function at that point is

$$\frac{dy}{dx} = -\frac{3}{\pi} \approx -0.95.$$

That looks about right:



Intervals. An *interval* of the real numbers $\mathbb{R}$ is the set of all numbers between two given numbers. They may or may not contain the endpoints. The following are examples of intervals:

$$\begin{aligned}(-4,6) &= \{x \in \mathbb{R} : -4 < x < 6\} \\ (-4,6] &= \{x \in \mathbb{R} : -4 < x \leq 6\} \\ [-4,6) &= \{x \in \mathbb{R} : -4 \leq x < 6\} \\ [-4,6] &= \{x \in \mathbb{R} : -4 \leq x \leq 6\}\,.\end{aligned}$$

If an interval includes both of its endpoints, as with $[-4,6]$, it is called *closed*. If it includes neither of its endpoints, as with $(-4,6)$, it is *open*. Intervals can also be "infinite", as in the following examples:

$$\begin{aligned}(-\infty,5) &= \{x \in \mathbb{R} : x < 5\} \\ (-\infty,5] &= \{x \in \mathbb{R} : x \leq 5\} \\ (5,\infty) &= \{x \in \mathbb{R} : x > 5\} \\ [5,\infty) &= \{x \in \mathbb{R} : x \geq 5\} \\ (-\infty,\infty) &= \mathbb{R}.\end{aligned}$$

The final example shows that $\mathbb{R}$ itself is considered to be an interval. For technical reasons, $\mathbb{R}$ is considered to be both open and closed. The previous, non-infinite intervals are called *bounded intervals*.

Extrema. Let f be a function defined on an interval I, and let c be an element of I. Then

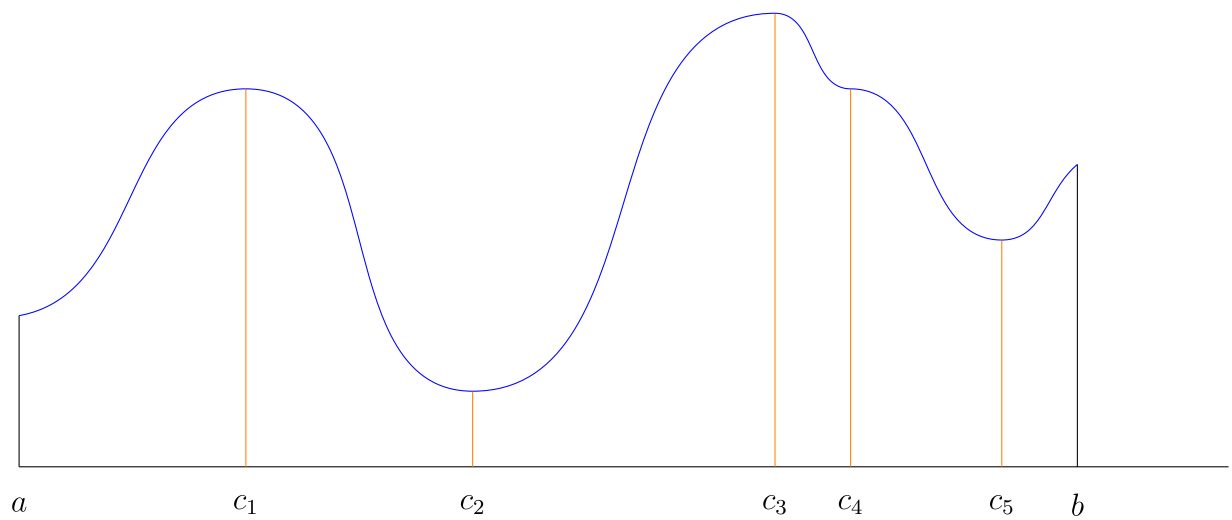
- f has a *minimum* at $c \in I$ if $f(c) \leq f(x)$ for all $x \in I$.

- f has a *maximum* at $c \in I$ if $f(c) \geq f(x)$ for all $x \in I$.

The minima and maxima are called *extrema*. Minima and maxima like these are sometimes called *global* minima and maxima to distinguish them from the following:

- f has a *relative (or local) minimum* at $c \in I$ if the exists an open interval containing c and contained in I on which f has a minimum at c.

- f has a *relative (or local) maximum* at $c \in I$ if the exists an open interval containing c and contained in I on which f has a maximum at c.



Graph of a function f.

c_1	-	local maximum
c_2	-	minimum
c_3	-	maximum
c_4	-	$f'(c_4) = 0$ but neither local minimum nor local maximum
c_5	-	local minimum.

The endpoints a and b are not (global) minima or maxima, and they can't be local minima or maxima since there is no open interval containing these points inside the interval $[a, b]$.

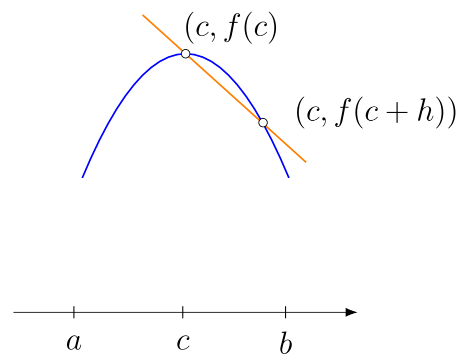
Warning. Here is a fine point: notice the less than or equal signs in the definition of minima and maxima rather than strict inequalities. This means. for instance, that for a constant function, *every* point is both a minimum and a maximum.

Two main results.

For the first result, note that the slope of the graph of a function at a local minimum or maximum must be 0. That's the content of the following theorem:

Theorem 1. If f is differentiable at c and f has a local minimum or maximum at c, then $f'(c) = 0$.

Proof. We will consider the case of a local maximum. The case of a local minimum is similar. So suppose f has a local maximum at c as pictured below:



We have $f(x) \geq f(c)$ for all x in the interval (a, b). Consider

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

If $h > 0$, then $f(c+h) \leq f(c)$, so

$$\frac{f(c+h) - f(c)}{h} \leq 0.$$

It follows that for the right-hand limit

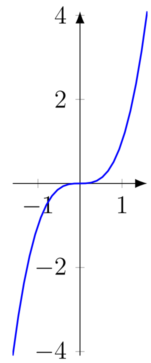
$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0.$$

Reasoning similarly about the case where $h < 0$, we get for the left-hand limit

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0.$$

(In this case, we'll have that both the numerator and denominator will be negative.) However, since $f'(c)$ exists, the right-hand and left-hand limits must be equal. So the only possibility is that they are both 0, which implies $f'(c) = 0$, as required. ☐

Warning: The converse of the above theorem does not necessarily hold. A function can flatten out, i.e., it can have slope 0, at a point that is not a local minimum or maximum. For example, consider the point $x = 0$ for the function $f(x) = x^3$:



Graph of $f(x) = x^3$.

Theorem 2. (The extreme value theorem, EVT) If f is continuous on a closed bounded interval $[a, b]$, then f has a (global) minimum and maximum on that interval.

Proof. Math 112. ☐

Note that the EVT assumes a closed interval. For instance, the function $f(x) = 1/x$ is continuous on the open interval $(0, 1)$ but is unbounded, hence has no maximum value. For that matter, the function $g(x) = x$ has no maximum or minimum on the open interval $(0, 1)$ even though the function is bounded on that interval. In contrast, the function $g(x)$ does have a maximum, 1, and a minimum, 0, on the *closed* interval $[0, 1]$.

Week 6, Friday: Optimization.

Here are the two most basic theorems for optimization theory:

Theorem 1. If f is differentiable at c and f has a local minimum or maximum at c, then $f'(c) = 0$.

Proof. See the notes from last time. □

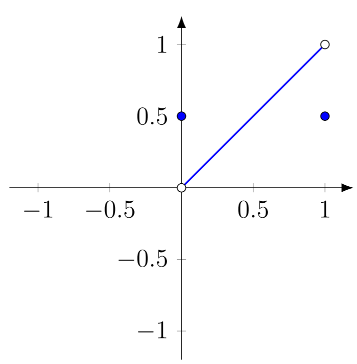
Theorem 2. (The extreme value theorem, EVT) If f is continuous on a closed bounded interval $[a, b]$, then f has a (global) minimum and maximum on that interval.

Proof. Math 112. □

The extreme value theorem can fail if either f is not continuous or the interval is not closed and bounded. Some examples demonstrating this:

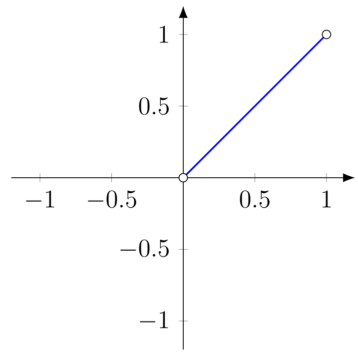
Example 1. Non-continuous $f : [0, 1] \to \mathbb{R}$ with no minimum or maximum:

$$f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ \frac{1}{2} & \text{if } x = \pm 1. \end{cases}$$



Graph of f.

Example 2. The function $g(x) = x$ is continuous on the open interval $(0, 1)$ but has no minimum or maximum on that interval:



Graph of g.

Example 3. The function $h(x) = 1/x$ is continuous on the open interval $(0, 1)$, but has no maximum or minimum that interval.

Example 4. The function $k(x) = x$ is continuous on the unbounded interval $(-\infty, \infty)$, but has no minimum or maximum on that interval.

Procedure for optimization. Given a continuous function f on a closed bounded interval $[a, b]$, to find the (global) minima and maxima for f:

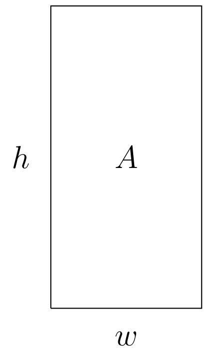
1. Collect the points in $[a, b]$ at which f is not differentiable.

2. Add the points at which the derivative of f is 0.

3. Add the endpoints, a and b.

The points satisfying 1 or 2 are called *critical points*. Evaluate f at the critical points and the endpoints. The smallest value will give the minimum and the largest will give the maximum.

Example 5. Suppose a farmer has ℓ feet of fence and wants to make a rectangular enclosure of the maximal area. What should the dimensions of the rectangle be? Similar to related rates problems, the first step in solving this problem is to draw a picture and label the relevant features:



We are given that the total length of fence is

$$\ell = 2w + 2h. \tag{17.1}$$

We are trying to find w and h in order to maximize the area

$$A = wh. \tag{17.2}$$

Since this is one-variable calculus, we need to write A as a function of one variable. We do this by solving for h in equation (17.1):

$$h = \frac{\ell}{2} - w.$$

Subsitute into equation (17.2):

$$A = \left(\frac{\ell}{2} - w\right) w = \frac{1}{2}\ell w - w^2. \tag{17.3}$$

We have now written A as a function of just one variable, w. Note that ℓ was fixed at the beginning of the problem—it's a constant.

Here is an important step that's easy to forget: to use our optimization technique, we should check the interval that constrains our variable, w. Ideally, it's a closed and bounded interval so that the EVT applies. In our case,

$$w \in [0, \ell/2].$$

So we can now apply our technique:

$$\frac{dA}{dw} = \frac{1}{2}\ell - 2w = 0 \quad \Longleftrightarrow \quad w = \frac{\ell}{4}.$$

So there is one critical point, $w = \ell/4$. Evaluate A at this point and at the endpoints of our interval using equation (17.3):

$$A\left(\frac{\ell}{4}\right) = \left(\frac{\ell}{2} - \frac{\ell}{4}\right) \cdot \frac{\ell}{4} = \frac{\ell^2}{16}$$
$$A(0) = A\left(\frac{\ell}{2}\right) = 0.$$

So the maximum area occurs when $w = \ell/4$. From equation (17.1) is then follows that $h = \ell/4$, too. So the way to maximize the area is to make a square.

Week 7, Wednesday: Optimization.

We went over problems on the midterm for most of this class. The solutions are posted at our class homepage.

We did have time for one optimization problem. Recall the general method. If we are lucky, we have a continuous function f on a closed bounded interval $[a, b]$. In that case, the extreme value theorem (EVT) assures the existence of a minimum and a maximum on the interval. We then compute the critical points x in $[a, b]$, i.e., those points where the derivative does not exist or is 0. We evaluate f at the critical points, and we also compute $f(a)$ and $f(b)$. We then pick the smallest and largest values.

Example. Let $f(x) = x^3 - x$. Find the minima and maxima for f on the interval $[-1, 2]$.

SOLUTION: Follow the procedure outlined above. First collect the points:

1. f is differentiable at all points in $[-1, 2]$ (in fact, f is differentiable at all points in $\mathbb{R}$. So this step gives us no interesting points to check.

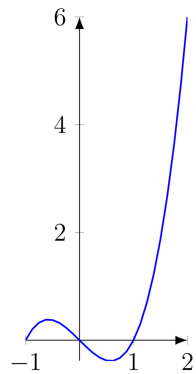
2. We have
$$f'(x) = 3x^2 - 1 = 0 \quad \text{if and only if} \quad x = \pm\sqrt{\frac{1}{3}}.$$

3. The endpoints of the interval are -1 and 2.

So we look for the extrema of f among the points $\pm\sqrt{1/3}$, -1, and 2. Evaluate f at these points:

$$f\left(-\sqrt{\frac{1}{3}}\right) = \frac{2}{3}\sqrt{\frac{1}{3}}, \quad f\left(\sqrt{\frac{1}{3}}\right) = -\frac{2}{3}\sqrt{\frac{1}{3}}, \quad f(-1) = 0, \quad f(2) = 6.$$

Thus, on the interval $[-1, 2]$, the minimum of f is $-2\sqrt{1/3}/3$, occurring at $\sqrt{1/3}$, and the maximum is 6, occuring at the endpoint, 2.



Graph of $f(x) = x^3 - x$ on the interval $[-1, 2]$.

While we have this picture, let's check that the derivative actually is giving the slope. We computed

$$f'(x) = 3x^2 - 1.$$

Therefore, the graph should be sloped upwards if and only if

$$f'(x) > 0 \quad \Rightarrow \quad 3x^2 - 1 > 0 \quad \Rightarrow \quad x^2 > \frac{1}{3} \quad \Rightarrow \quad x > \sqrt{\frac{1}{3}} \quad \text{or} \quad x < -\sqrt{\frac{1}{3}}.$$

Week 7, Friday: Optimization.

A couple more optimization problems. The first illustrates a case where the extreme value theorem does not immediately apply because the interval on which the function is defined is not closed.

Example 1. Consider the function

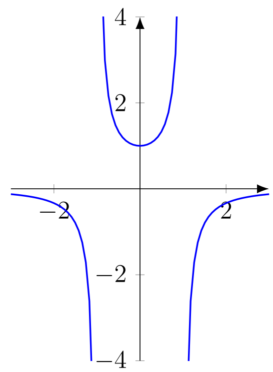
$$f(x) = \frac{1}{1 - x^2}$$

on the interval $(-1, 1)$. First note that f is continuous on $(-1, 1)$. However, our procedure does not apply, unfortunately, since $(-1, 1)$ is not a closed interval. So we need to be creative. First, it's clear the function blows up where $x = \pm 1$. In fact, as $x \to -1$ or $x \to 1$ from points inside the interval, the function f takes off to $+\infty$. Therefore, f has no maximum value on $(-1, 1)$. We can look for a minimum, though. Since f is differentiable on $(-1, 1)$, we know that f' will be 0 at any minimum (it needs to flatten out at these points). Computing the derivative of f using the quotient rule gives

$$f'(x) = \frac{2x}{(1 - x^2)^2}.$$

We have $f'(x) = 0$ only at the point $x = 0$.

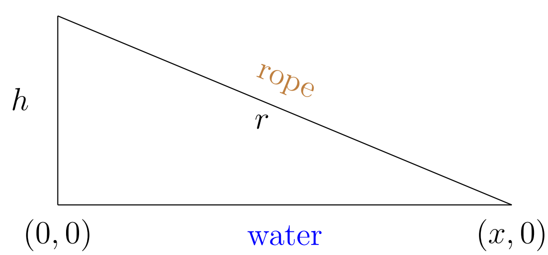
So far, we can only conclude that f has a *local* minimum at $x = 0$. However, note that for $x \in (-1, 1)$ we have that $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$. This means that f is sloped downwards for $x < 0$ and sloped upwards for $x > 0$. This guarantees that $x = 0$ is a global minimum on the interval $(-1, 1)$.



Graph of $f(x) = \frac{1}{1-x^2}$.

One more related rates problem. Suppose you are standing on a dock pulling in a boat attached to a rope. If you pull the rope in at a constant rate, how does the speed at which the boat approaches the dock change?

SOLUTION: The relevant picture is:



The height of the point where the rope is being pulled is h feet above the water. We will assume that h is constant. The boat is x feet away from the dock, and the rope has length r. We are given that

$$\frac{dr}{dt} = -k = \text{ constant.}$$

We take $k > 0$, so the minus sign tells us that the rope is getting shorter over time. We are interested in the rate of change of x, i.e., in dx/dt. The equation relating the variables is

$$x^2 + h^2 = r^2.$$

Take derivatives with respect to time, remembering that h is constant:

$$2x\frac{dx}{dt} = 2r\frac{dr}{dt},$$

or

$$x\frac{dx}{dt} = r\frac{dr}{dt}.$$

Since $dr/dt = -k$, we get

$$x\frac{dx}{dt} = -kr.$$

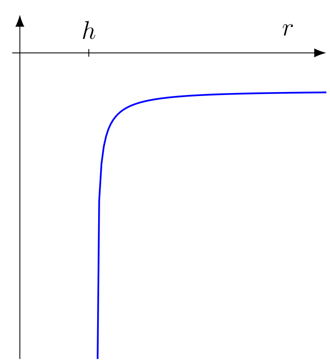
To get a solution in terms of r, we solve $x^2 + h^2 = r^2$ for x and substitute:

$$\frac{dx}{dt} = -k\frac{r}{\sqrt{r^2 - h^2}}.$$

So it is clear the speed of the boat is not constant. In fact, as time goes on, r approaches h, so the numerator is bounded around h with the denominator goes to 0, so

$$\lim_{r\to h}\frac{dx}{dt} = \lim_{r\to h} -k\frac{r}{\sqrt{r^2 - h^2}} = -\infty.$$

The graph of dx/dt as a function of r for $r > h$ looks something like this:



Graph of $\frac{dx}{dt}$.

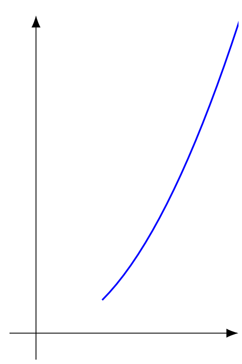
Week 8, Monday: Curve sketching; second derivative test.

Second derivatives. If f is differentiable, then its derivative $f'(x)$ is a function of x and it makes sense to take its derivative. The derivative of $f'(x)$ is called the *second derivative of* f and denoted $f''(x)$.

Example. If $f(x) = x^4 - 3x^2 + 2$, then $f'(x) = 4x^3 - 6x$, and $f''(x) = 12x^2 - 6$.

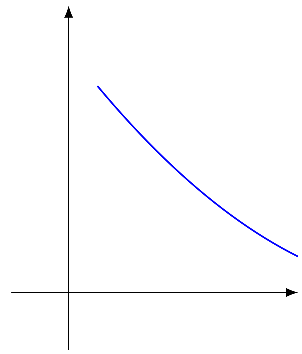
Just as $f'(x)$ gives the rate of change or slope of f, the second derivative, $f''(x)$, gives the rate of change or slope of $f'(x)$. If $f(t)$ gives the distance a particle has traveled after times t, the $f'(t)$ is the speed of the particle at time t, and $f''(t)$ is the acceleration of the particle at time t.

From the graph of f, we can tell where $f'(x) > -1$ by looking at the places where the slope of f is positive. Imagine what the graph of f looks like at places where $f''(x) > 0$. The slope of the graph would have to be increasing, which is called being *concave up*. That could happen at places where the graph is sloped upwards but getting steeper, as pictured below:



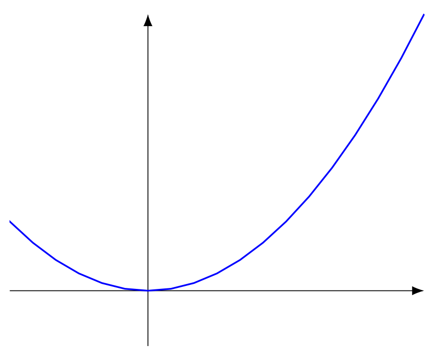
Graph of a function f with $f'' > 0$.

The slope of f can also be increasing if it goes from having a negative slope to having a less negative slope, as pictured below:



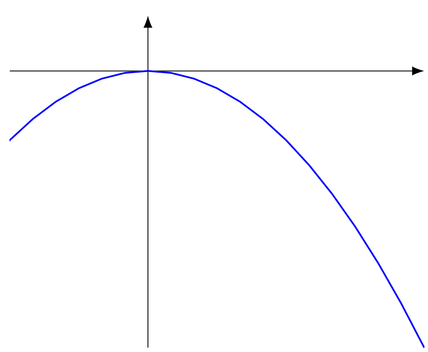
Graph of a function f where $f'' > 0$.

Another possibility is if the slope of f is changing from negative to positive, as pictured below:



Graph of a function f with $f'' > 0$.

Similarly, a graph with $f'' < 0$ is *concave down*:



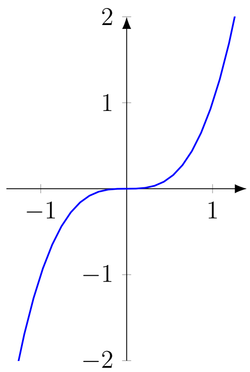
Graph of a function f with $f'' < 0$.

In summary,

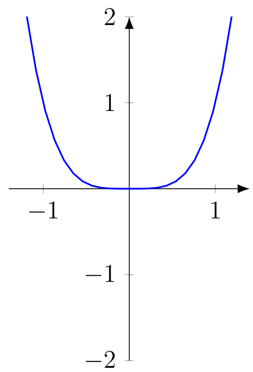
$$f''(x) > 0 \implies \text{concave up}$$
$$f''(x) < 0 \implies \text{concave down.}$$

The second derivative test. Suppose that f is a function whose first and second derivatives exist, and say that $f'(c) = 0$. There are three possibilities:

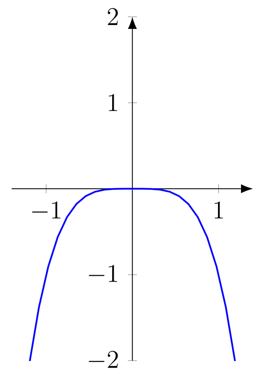
- $f'(c) = 0$ and $f''(c) > 0$. This means that f flattens out at c and is concave up around c. Therefore, c is a **local minimum** of f. (For example, consider $f(x) = x^2$ at $c = 0$.)
- $f'(c) = 0$ and $f''(c) < 0$. This means that f flattens out at c and is concave down around c. Therefore, c is a **local maximum** of f. (For example, consider $f(x) = -x^2$ at $c = 0$.)
- $f'(c) = 0$ and $f''(c) = 0$. This case is more complicated. For example, each of the following functions has $f'(0) = f''(0) = 0$:



$f(x) = x^3$.



$f(x) = x^4$.



$f(x) = -x^4$.

There are three subcases:

– $f'(c) = 0 = f''(c) = 0$, and $f'(x)$ is positive except at c in an interval containing c. Then f has a local minimum at c.

– $f'(c) = 0 = f''(c) = 0$, and $f'(x)$ is negative except at c in an interval containing c. Then f has a local maximum at c.

– $f'(c) = 0 = f''(c) = 0$, and $f'(x)$ changes sign at c. In that case, we say that c is a *point of inflection for* f.

Curve sketching using derivatives. How can one approximate the graph of a function without using a computer? Here is a checklist:

1. Calculate where $f'(x) = 0$ and where $f'(x)$ does not exist (i.e., find the *critical points* of f), and evaluate the function at these points. Draw these points in your graph.
2. Determine the sign of f' between the critical points (in order to figure out how the slope of f changes).
3. Find the *zeros* of f, i.e., those points where $f(x) = 0$. Draw these points in your graph.
4. What happens to $f(x)$ as $x \to \infty$ and as $x \to -\infty$? (Are there horizontal asymptotes?)
5. Find places where the function "blows up", i.e., find any vertical asymptotes. If you find a vertical asymptote, how does f behave on either side of it?
6. To determine concavity, you can use the second derivative: $f''(x) > 0 \Rightarrow$ concave up, and $f''(x) < 0 \Rightarrow$ concave down.

Example 1. $f(x) = x^4 - 2x^2$.

1. Critical points:

$$\begin{aligned} f'(x) = 4x^3 - 4x = 0 \quad &\Longleftrightarrow \quad x^3 = x \\ &\Longleftrightarrow \quad x = 0 \quad \text{or} \quad x^2 = 1 \\ &\Longleftrightarrow \quad x = -1, 0, 1. \end{aligned}$$

   We have $f(-1) = -1$, $f(0) = 0$, and $f(1) = -1$. So we plot the points $(-1, -1)$, $(0, 0)$, and $(1, -1)$.

2. The sign of f' between critical points:

slope of f:	down		up		down		up
f':	$-$		$+$		$-$		$+$
		-1		0		1	

3. Zeros of f:

$$f(x) = x^4 - 4x = 0 \quad \Longleftrightarrow \quad x = -\sqrt{2}, 0, \sqrt{2}.$$

4. Horizontal asymptotes

$$\lim_{x\to\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \infty.$$

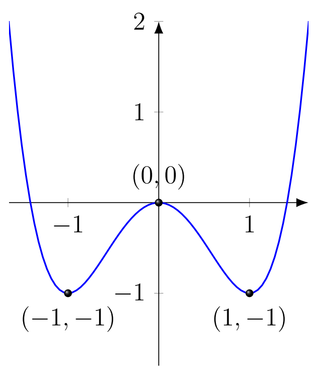
   No horizontal asymptotes.

5. No vertical asymptotes.

6. Concavity:

$$\begin{aligned} f''(x) = 12x^2 - 4 > 0 \quad &\iff \quad x^2 > \frac{1}{3} \\ &\iff \quad x > \sqrt{\frac{1}{3}} \quad \text{or} \quad x < -\sqrt{\frac{1}{3}}. \end{aligned}$$

In particular, at the critical points, we have $f''(-1) = f''(1) = 8 > 0$ (concave up, hence, local minima), and $f(0) = -4 < 0$ (concave down, hence, a local maximum). We have $f''(x) = 0$ at $x = \pm\sqrt{1/3}$, and these are inflection points.



Graph of $f(x) = x^4 - 2x^2$.

Example 2. $f(x) = x^4 - 4x^3 = x^3(x-4)$.

1. Critical points:

$$\begin{aligned} f'(x) = 4x^3 - 12x^2 = 0 \quad &\iff \quad x^3 = 3x \\ &\iff \quad x = 0, 3 \end{aligned}$$

   We have $f(0) = 0$, and $f(3) = -27$. So we plot the points $(0, 0)$, and $(3, -27)$.

2. The sign of f' between critical points:

slope of f:	down	down	up
f':	$-$	$-$	$+$
		0	3

3. Zeros of f:

$$f(x) = x^4 - 4x^3 = 0 \iff x = 0, 4.$$

4. Horizontal asymptotes

$$\lim_{x\to\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \infty.$$

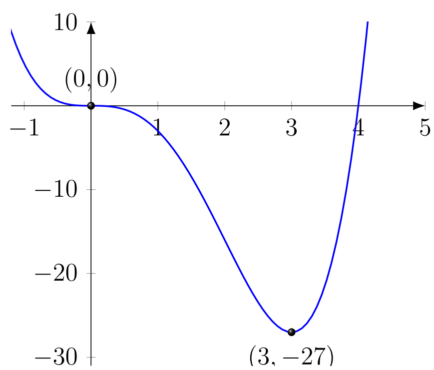
   No horizontal asymptotes.

5. No vertical asymptotes.

6. Concavity:

$$\begin{aligned} f''(x) = 12x^2 - 24x > 0 &\iff x^2 - 2x > 0 \\ &\iff x(x-2) > 0 \end{aligned}$$

The points $x = 0$ and $x = 2$ are inflection points.



Graph of $f(x) = x^4 - 4x^3$.

Example 3. $f(x) = \dfrac{x^2 - 2x + 2}{x - 1}$.

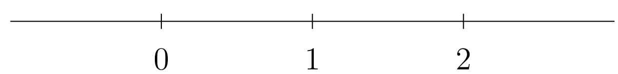
1. Critical points:

$$f'(x) = \frac{x(x-2)}{(x-1)^2} = 0 \quad \Longleftrightarrow \quad x = 0, 2 \quad \text{and undefined at } x = 1.$$

   We have $f(0) = -2$, and $f(2) = 2$. So we plot the points $(0, -2)$, and $(2, 2)$.

2. The sign of f' between critical points:

slope of f:	up	down	down	up
f':	$+$	$-$	$-$	$+$



3. Zeros of f:

$$f(x) = \frac{x^2 - 2x + 2}{x - 1} = 0 \quad \Longrightarrow \quad x^2 - 2x + 2 = 0.$$

   Using the quadratic equation, we see there are no real zeros:

$$x^2 - 2x + 2 = 0 \quad \Longrightarrow \quad x = \frac{2 \pm \sqrt{4 - 8}}{2}.$$

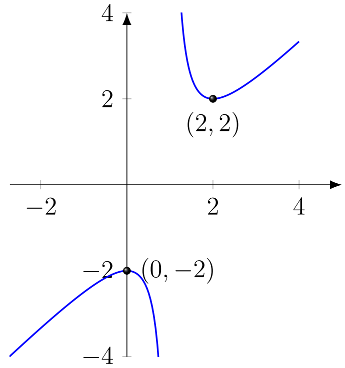
4. Horizontal asymptotes

$$\lim_{x \to \infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = -\infty.$$

   No horizontal asymptotes. However, note that for large x, we have $f(x) \approx \frac{x^2}{x} = x$. So the line $y = x$ is a kind of asymptote.

5. Vertical asymptote at $x = 1$.

6. Concavity: skipped since the second derivative is kind of messy.



Graph of $f(x) = \dfrac{x^2 - 2x + 2}{x - 1}$.

Week 8, Wednesday: Least upper bounds, greatest lower bounds. Estimating areas.

Our next goal is to define the integral of a function. If f is a nonnegative function defined on a closed bounded interval $[a, b]$, the integral $\int_a^b f$ will be used to define the area between the graph of f and the x-axis.

Upper and lower bounds. Let X be any subset of the real numbers, $\mathbb{R}$. An *upper bound* for X is *any* real number B that is at least as big as all the numbers in X, i.e.,

$$x \leq B \quad \text{for all } x \in X.$$

Similarly, a An *lower bound* for X is any real number b that is no greater than any numbers in X, i.e.,

$$b \leq x \quad \text{for all } x \in X.$$

Finally, the set X is simply called *bounded*, if it has both an upper and lower bound.

Examples.

1. Let $X = \{-2, 7, 9\}$, a set consisting of 3 real numbers. Then every number greater than or equal to 9 is an upper bound for X, and every number less than or equal to -2 is a lower bound. So 9, 9.1, and 27 are examples of upper bounds for X and -7, -7.234, and $-\pi$ are examples of lower bounds for X. The set X has upper and lower bounds, so X is bounded.

2. Let $Y = [3, 100) = \{x \in \mathbb{R} : 3 \leq x \leq 100\}$, a half-open, half-closed interval. The set Y is bounded: any number greater than or equal to 100 is an upper bound and any number less than or equal to 3 is a lower bound.

3. Let $Z = [0, \infty)$, the set of nonnegative real numbers. Then Z has no upper bounds, and every nonpositive number is a lower bound. The set Z is not bounded. It is only bounded below.

4. The set $\mathbb{R}$ of all real numbers has no upper bound and no lower bound. The same goes for the integers, $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.

Least upper bounds and greatest lower bounds. In general: if X has an upper bound, then it has infinitely many upper bounds, and if X has a lower bound, then it has infinitely many lower bounds. There is a property of the real numbers that say this: if a set X has an upper bound, then it has a *least upper bound*. The least upper bound, denoted $\text{lub}(X)$, it characterized by two properties: (i) it is an upper bound for X, and (ii) it is less than or equal to every upper bound for X.

Similarly, if X has a lower bound, then it has a *greatest lower bound*. It is denoted by $\text{glb}(X)$ had is characterized by the two properties, (i) is is a lower bound for X, and (ii) is is greater than or equal to every lower bound for X.

Thus, in some sense, $\text{lub}(X)$ and $\text{glb}(X)$ are the "best" upper and lower bounds, respectively, for X, provided they exist.

Example.

1. If $X = \{-2, 7, 9\}$, then $\text{lub}(X) = 9$ and $\text{glb}(X) = -2$.
2. If $Y = [3, 100)$, then $\text{lub}(X) = 100$ and $\text{glb}(X) = 3$.
3. If $Z = [0, \infty)$, then $\text{lub}(X)$ does not exist, and $\text{glb}(X) = 0$. Similarly, $\text{lub}((-\infty, 0]) = 0$ and $\text{glb}((-\infty, 0])$ does not exist.
4. The set $\mathbb{R}$ of all real numbers has neither a least upper bound nor a greatest lower bound. The same holds for the set of integers $\mathbb{Z}$.
5. Let
$$A = \left\{\frac{1}{n} : n = 1, 2, 3, \dots\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}.$$
Then $\text{lub}(A) = 1$, and $\text{glb}(A) = 0$.
6. Let
$$B = \left\{\frac{n}{n+1} : n = 1, 2, 3, \dots\right\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right\}.$$
Then $\text{lub}(B) = 1$, and $\text{glb}(A) = 1/2$.

NOTE WELL: Even if the least upper and greatest lower bounds for a set exist, they need not be in the set. For example, consider the set

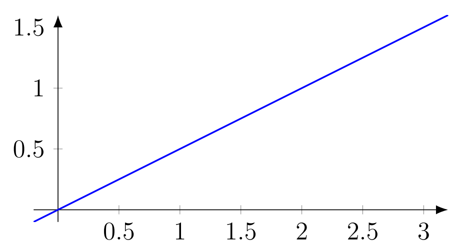
$$X = (0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}.$$

Then lub(X) = 1 and glb(X) = 0. However, neither 0 nor 1 is in $(0, 1)$. This interval, by definition does not contain its endpoints. It's hard to appreciate the importance of this fact now, but lub and glb are as important in defining the integral as limits were to defining the derivative. Recall that with derivatives, we need to find out what value the function of h

$$\frac{f(c+h) - f(c)}{h}$$

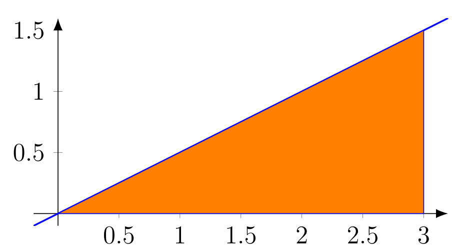
"should" have when h is 0 even though the function is actually undefined at that point. In a similar way, lub(X) is the number that should be the "biggest element of X" even though X may not have a biggest element. (For instance, the set $(0, 1)$ has no largest element: 1 is not in $(0, 1)$, and given any element in $x \in (0, 1)$ there is another element in $y \in (0, 1)$ such that $x < y$—just take y to be the number that is halfway between x and 1.)

Integration. We start off with a relatively simple example. Consider the function $f(x) = x/2$ on the interval $[0, 3]$:



Graph of $f(x) = \dfrac{x}{2}$.

We are interested in computing the area of the colored region that is under the graph:



Graph of $f(x) = \dfrac{x}{2}$.

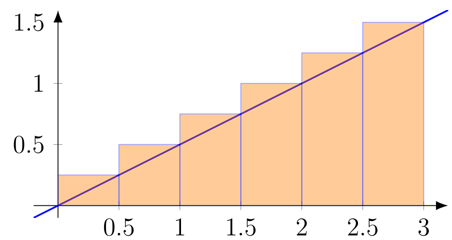
Since this is a triangle, its area is

$$\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 3 \times \frac{3}{2} = \frac{9}{4}.$$

To motivate the definition of the integral, we are going to calculate this area in a more difficult way (but which has the advantage of generalizing to much more complicated functions).

The idea of integration is to compute area using only rectangles: give any region, divide the region up into rectangles as close as you can, then estimate the region by adding up the areas of the approximating rectangles. Of course, if the region is curved, you can never divide it up exactly into a finite number of rectangles, although, if you are willing to use very tiny rectangles, you can hope to estimate the area as close as you'd like. However, if you want the precise area, you can see that this rectangle method is going to lead to a limiting process of some sort.

Below, we create an overestimation of the area of the triangle:



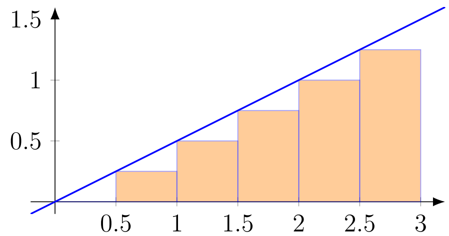
Graph of $f(x) = \dfrac{x}{2}$.

To estimate the area, we add the areas of these rectangles. The base of each rectangle is $1/2$, What about the height? Since the function is $f(x) = x/2$, to find the height, we evaluate f at the right-hand endpoint of the base of each rectangle. The result is

$$\begin{aligned}
E &= \left(\frac{1}{2}\cdot\frac{1}{4}\right) + \left(\frac{1}{2}\cdot\frac{1}{2}\right) + \left(\frac{1}{2}\cdot\frac{3}{4}\right) + \left(\frac{1}{2}\cdot 1\right) + \left(\frac{1}{2}\cdot\frac{5}{4}\right) + \left(\frac{1}{2}\cdot\frac{3}{2}\right) \\
&= \frac{1}{2}\left(\frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1 + \frac{5}{4} + \frac{3}{2}\right) \\
&= \frac{1}{2}\left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{4}{4} + \frac{5}{4} + \frac{6}{4}\right) \\
&= \frac{1}{2}\cdot\frac{1}{4}\left(1+2+3+4+5+6\right) \\
&= \frac{21}{8} = 2.625.
\end{aligned}$$

This overestimates the actual area of the triangle, $9/4 = 2.25$.

We now create an underestimation of the area of the triangle:



Graph of $f(x) = \dfrac{x}{2}$.

The heights of the rectangles are now given by evaluating $f(x) = x/2$ at the left-hand endpoint of the base of each rectangle:

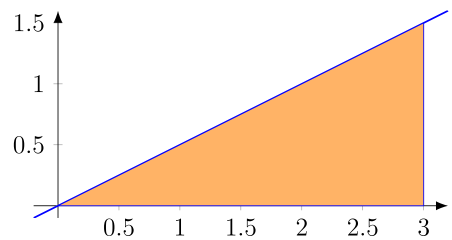
$$\begin{aligned}
E' &= \left(\frac{1}{2}\cdot 0\right) + \left(\frac{1}{2}\cdot\frac{1}{4}\right) + \left(\frac{1}{2}\cdot\frac{1}{2}\right) + \left(\frac{1}{2}\cdot\frac{3}{4}\right) + \left(\frac{1}{2}\cdot 1\right) + \left(\frac{1}{2}\cdot\frac{5}{4}\right) \\
&= \frac{1}{2}\left(0 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1 + \frac{5}{4}\right) \\
&= \frac{1}{2}\left(\frac{0}{4} + \frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{4}{4} + \frac{5}{4}\right) \\
&= \frac{1}{2}\cdot\frac{1}{4}\left(0+1+2+3+4+5\right) \\
&= \frac{15}{8} = 1.875.
\end{aligned}$$

This underestimates the actual area of the triangle, $9/4 = 2.25$.

To get better over- and underestimates, we can divide up the base into more pieces and thus create better-fitting rectangles.

Week 8, Friday: Definition of the integral.

Warm-up to the definition of the derivative. Last time, we were considering a way of finding upper- and lower-bounds for the area under the graph of $f(x) = x/2$ from $x = 0$ to $x = 3$.



Graph of $f(x) = \dfrac{x}{2}$.

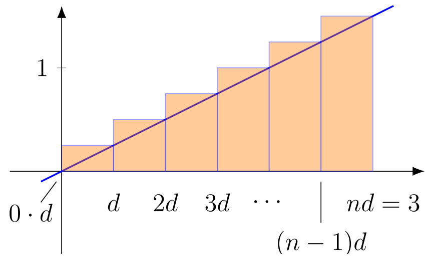
Of course, since the region is a triangle, we can easily compute its area with the usual formula: $(1/2) \times$ base $\times$ height $= 9/4 = 2.25$.

Last time, we tried approximating the area by dividing the interval $[0, 3]$ into 6 equal parts, then creating rectangles over each of these parts. The sum of the areas of the rectangles then gave an estimate of the area of the triangle. We want to generalize that approach now. Instead of dividing the region into 6 equal-sized pieces, let's divide it into n equal-sized pieces where n is any positive integer. Each piece would then have a length we will denote by d. So,

$$d := \frac{3-0}{n} = \frac{3}{n}.$$

The case $n = 6$ gives $d = 3/6 = 1/2$, as in our example.

The figure below attempts to illustrate the result of dividing the base of the triangle up into n parts (even though there are only 6 try to imagine a lot more parts signified by the dots, $\cdots$. (The term $(n-1)d$ appears below the other marks just because it wouldn't fit.)



Graph of $f(x) = \dfrac{x}{2}$.

Our task now is to add up the areas of the n rectangles (base$\times$ height). Each base has length $d = 3/n$. The rectangles have different heights. The height is determined by the right-hand endpoint of the base of the rectangle. Say that point is kd for some k. The rectangle's height is determined by the height of the graph, which would be $f(kd) = kd/2$. Hence, the area for this rectangle would be base$\times$height $= d \cdot kd/2$. Adding up the areas, we get

$$\text{sum of areas} = d \cdot \frac{d}{2} + d \cdot \frac{2d}{2} + d \cdot \frac{3d}{2} + \cdot + d \cdot \frac{(nd)}{2}$$

$$= \frac{d^2}{2} \cdot (1 + 2 + 3 + \cdots + n).$$

It turns out there is a useful closed formula for $1 + 2 + \cdots + n$. So we will pause in our calculation of the area to talk about that formula.

Lemma. For each $n > 0$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

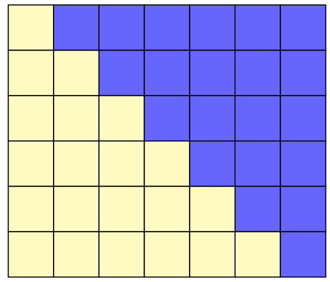
Proof. As a warm-up, we do the case $n = 6$. Here is the trick:

$$\begin{array}{rc} & 1+2+3+4+5+6 \\ + & 6+5+4+3+2+1 \\ \hline & 7+7+7+7+7+7 \end{array} \quad = 6 \cdot 7$$

Adding the sum twice gives $6 \cdot 7 = 42$. Divide by two to get the sum:

$$1 + 2 + 3 + 4 + 5 + 6 = \frac{6 \cdot 7}{2} = 21.$$

Here is a "proof by picture":



The 7×7 square contains our sum twice—once in yellow and once in blue. The proof clearly generalizes:

$$
\begin{array}{rccccccccccc}
 & 1 & + & 2 & + & \cdots & + & (n-1) & + & n & & \\
+ & n & + & (n-1) & + & \cdots & + & 2 & + & 1 & & \\
\hline
 & (n+1) & + & (n+1) & + & \cdots & + & (n+1) & + & (n+1) & & = n \cdot (n+1)
\end{array}
$$

Divide by two to get the general sum formula:

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

□

We now return to adding the areas of the rectangles:

$$
\begin{aligned}
\text{sum of areas} &= d\cdot\frac{d}{2} + d\cdot\frac{2d}{2} + d\cdot\frac{3d}{2} + \cdot + d\cdot\frac{(nd)}{2} \\
&= \frac{d^2}{2}\cdot(1+2+3+\cdots+n) \\
&= \frac{d^2}{2}\cdot\frac{n(n+1)}{2} \\
&= \frac{n(n+1)}{4}d^2 \\
&= \frac{n(n+1)}{4}\left(\frac{3}{n}\right)^2 \\
&= \frac{9}{4}\frac{(n+1)}{n} \\
&= \frac{9}{4}\left(1+\frac{1}{n}\right).
\end{aligned}
$$

For each n, we get an overestimate of the area of the triangle. Here is a set listing these overestimates for $n = 1, 2, 3, \ldots$:

$$X = \left\{\left(\frac{9}{4}\cdot\frac{2}{1}\right), \left(\frac{9}{4}\cdot\frac{3}{2}\right), \left(\frac{9}{4}\cdot\frac{4}{3}\right), \ldots\right\}.$$

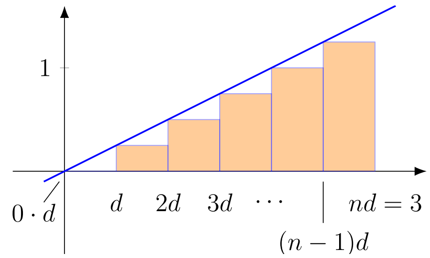
Notice that as n gets larger, the estimate gets smaller, each getting closer to the actually area, 9/4, but never reaching it. The number we are looking for, the actual area, is the greatest lower bound of X:

$$\text{area of triangle} = \text{glb}(X).$$

Note however that The argument we have given to this point does not assure us that the area is 9/4. We just know that the area is bounded above by 9/4. So we should really write:

$$\text{area of triangle} \leq \text{glb}(X) = \frac{9}{4}.$$

In order to be sure that the area is 9/4, we need to look at underestimates, too. So we now repeat the above argument, but this time with underestimates for the area:



Graph of $f(x) = \dfrac{x}{2}$.

This time the rectangles have heights determined by the left-hand endpoints of the subintervals:

$$\begin{aligned}
\text{sum of areas} &= d \cdot \frac{0 \cdot d}{2} + d \cdot \frac{1 \cdot d}{2} + d \cdot \frac{2d}{2} + \cdot + d \cdot \frac{(n-1)d}{2} \\
&= \frac{d^2}{2} \cdot (0 + 1 + 2 + \cdots + (n-1)) \\
&= \frac{d^2}{2} \cdot \frac{(n-1)n}{2} \\
&= \frac{(n-1)n}{4} d^2 \\
&= \frac{(n-1)n}{4} \left(\frac{3}{n}\right)^2 \\
&= \frac{9}{4} \frac{(n-1)}{n} \\
&= \frac{9}{4} \left(1 - \frac{1}{n}\right).
\end{aligned}$$

For each n, we get an underestimate of the area of the triangle. A set listing these underestimates for $n = 1, 2, 3, \ldots$:

$$Y = \left\{ \left(\frac{9}{4} \cdot \frac{0}{1}\right), \left(\frac{9}{4} \cdot \frac{1}{2}\right), \left(\frac{9}{4} \cdot \frac{2}{3}\right), \ldots \right\}.$$

As n gets larger, this time the estimate gets larger, each getting closer to the actually area, 9/4, but never reaching it. We have shown that

$$\frac{9}{4} = \mathrm{glb}(Y) \leq \text{area of triangle.}$$

Combining this with our early calculation involving overestimates of the area, we get

$$\frac{9}{4} = \mathrm{glb}(Y) \leq \text{area of triangle} \leq \mathrm{lub}(X) = \frac{9}{4}.$$

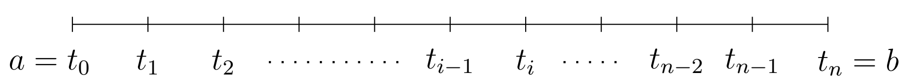
This finally proves that the area of the triangle is 9/4.

Week 9, Monday: Definition of the integral.

The integral. Now consider an arbitrary function f defined on an interval $[a, b]$. We would like to estimate the area under f by imitating what we just did with $f(x) = x/2$, above. Before, we divided the interval in question into n parts of equal length for convenience. In general, we allow division into *arbitrary length intervals.* To that end pick $n+1$ arbitrary points in the interval $[a, b]$, the first of which is a and the last of which is b:

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b.$$

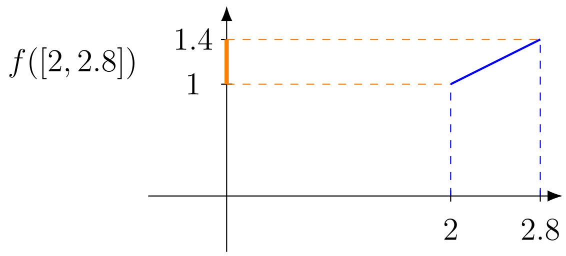
Here is a picture of the subdivision of $[a, b]$ into n parts (the dots connote an arbitrary number of tick marks):



So we have divided the interval into *subintervals.* The first subinterval is $[t_0, t_1] = [a, t_1]$. The second subinterval is $[t_1, t_2]$, and so on. In general, the *i-th subinterval* is $[t_{i-1}, t_i]$.
The values the function f takes on the i-th interval is denoted $f([t_{i-1}, t_i])$:

$$f([t_{i-1}, t_i]) = \{f(x) : t_{i-1} \leq x \leq t_i\}.$$

This set is called the *image of* $[t_{i-1}, t_i]$ under f. For instance, if the function is $f(x) = x/2$ and the i-th interval is $[2, 2.8]$, then $f([2, 2.8]) = [1, 1.4]$. We can picture the image as the set of y-values of the function as x varies along the i-th interval:



The image of the interval $[2, 2.8]$ under $f(x) = \dfrac{x}{2}$.

To estimate the area under f we create rectangles based on each subinterval. To overestimate the area, we will take the height of each rectangle to be the maximum value of the function on its interval, and to underestimate the area, we will take the height to be the minumum value. We introduce notation for these heights:

$$M_i = \operatorname{lub} f([t_{i-1}, t_i]) \quad \text{and} \quad m_i = \operatorname{glb} f([t_{i-1}, t_i]).$$

So M_i is the least upper bound of all function values at points in the i-th subinterval, and m_i is the greatest lower bound. This means

$$m_i \leq f(x) \leq M_i$$

for all x satistfying $t_{i-1} \leq x \leq t_i$.

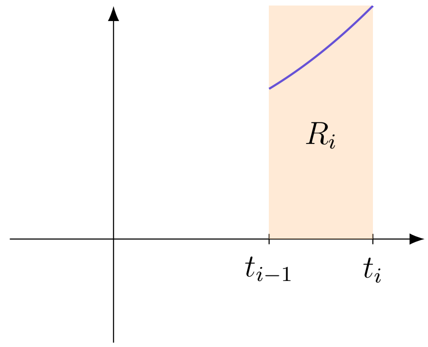
Time out for a technical point. Notice that we are using least upper bounds and greatest upper bounds instead of just taking the maximum value and the minimum value. There is a reason for that. Each subinterval is a closed bounded interval. If f is continuous, then the extreme value theorem guarantees that f has a maximum and a minimum value on the interval. However, if f is not continuous, it may not achieve its maximum or its minimum value on the interval. The set of function values will have a least upper bound and a greatest lower bound, however, as long as we assume the set of function values is bounded, which we will do from now on:

Assumption: From now on, we will assume that the set of values for f on the interval $[a, b]$ has is bounded (both above and below):

$$f([a, b]) = \{f(x) : a \leq x \leq b\},$$

is a bounded set of real numbers.

Back to defining the integral. We will next concentrate on creating an overestimate for the area under f. On the i-th subinterval, create a rectangle R_i with base $[t_{i-1}, t_i]$ and height M_i.



Rectangle on the i-th subinterval, overestimating the area.

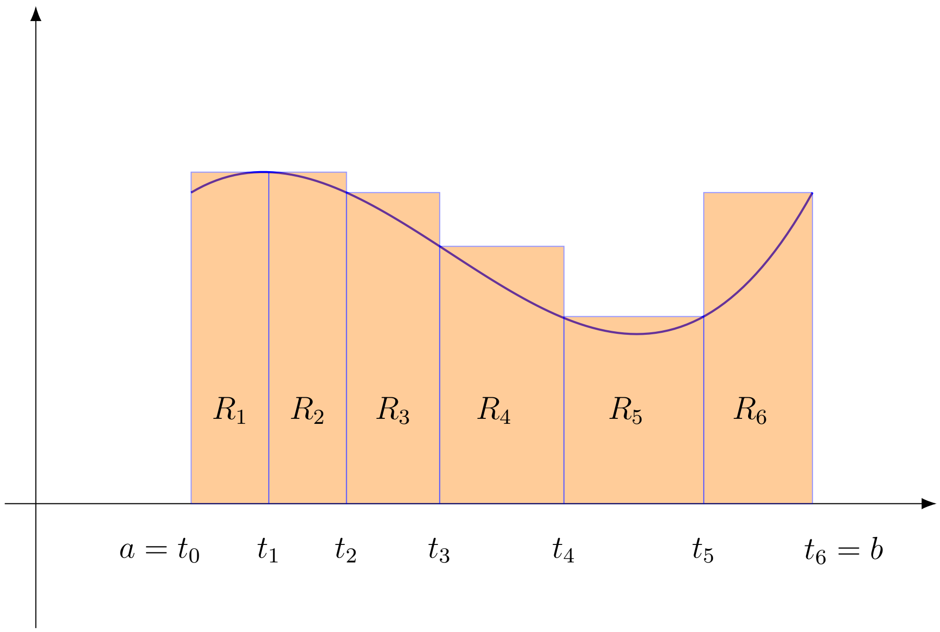
We have

$$\text{area}(R_i) = \text{height} \times \text{base} = M_i(t_i - t_{i-1}).$$

Adding up the areas of these rectangles gives an overestimate of the area called the *upper sum for* f *with respect to the partition* $P = \{t_0, t_1, \ldots, t_n\}$:

$$\begin{aligned} U(f,p) &:= \text{area}(R_1) + \text{area}(R_2) + \cdots + \text{area}(R_n) \\ &= M_1(t_1 - t_0) + M_2(t_2 - t_1) + \cdots + M_n(t_n - t_{n-1}) \\ &= \sum_{i=1}^{n} M_i(t_i - t_{i-1}). \end{aligned}$$

Below we illustrate the rectangles for an upper sum for some function f on a partition with 6 points:

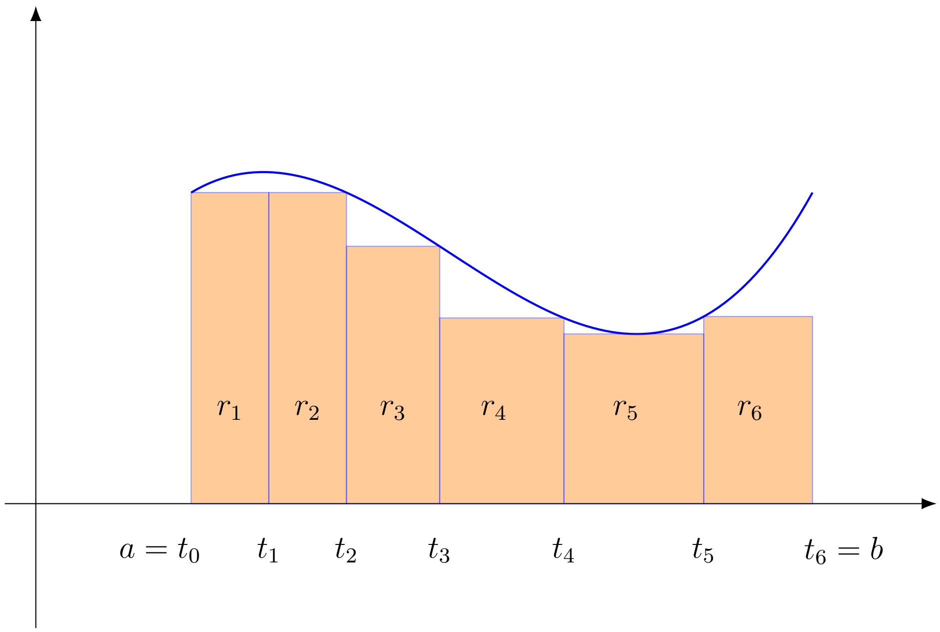


An upper sum $U(f, P)$ for some function f.

To get underestimates of the area, we repeat the above but now taking the heights of the rectangles to be the greatest lower bounds, m_i. Denote these rectangles by r_i. We define *lower sum for f with respect to the partition* P to be the sum of the areas of these rectangles:

$$\begin{aligned} L(f, p) &:= \text{area}(r_1) + \text{area}(r_2) + \cdots + \text{area}(r_n) \\ &= m_1(t_1 - t_0) + m_2(t_2 - t_1) + \cdots + m_n(t_n - t_{n-1}) \\ &= \sum_{i=1}^{n} m_i(t_i - t_{i-1}). \end{aligned}$$

The following picture shows the rectangles for the lower sum of some function f with respect to some partition P with 6 points:



A lower sum $L(f, P)$ for some function f.

For each partition P of $[a, b]$, we get an upper sum—an overestimate of the area under f. Let's make a set consisting of all possible upper sums as we vary the partition:

$$\{U(f, P) : P \text{ a partition of } [a, b]\}.$$

This will, in general, be a set consisting of an infinite number of numbers, on for each of the infinite number of partitions P, and each an overestimate of the area we want. For any given choice of function f, it will almost certainly not have a smallest element. However, it turns out it does have a greatest lower bound (see Math 112). This is in some sense or "best" overestimate". Officially, it is known as the *upper integral* for f:

$$U\int_a^b f := \text{glb}\,\{U(f, P) : P \text{ a partition of } [a, b]\}.$$

Similarly, we get one lower sum for each partition we choose, and we can consider the set of all possible underestimates of the area as the partition varies:

$$\{L(f, P) : P \text{ a partition of } [a, b]\}.$$

These are all underestimates, and in general there will be no greatest element in this set. However, it does have a least upper bound, and we define this to be the *lower* *integral* for f:

$$L\int_a^b f := \text{lub}\,\{L(f,P) : P \text{ a partition of } [a,b]\}\,.$$

So now we have a best overestimate and a best underestimate, and as might be expected (and proved in Math 112), we have

$$L\int_a^b f \leq U\int_a^b f.$$

For some unhappy functions, it turns out that these two numbers are not equal, and in that case we say f is *not integrable*. On the other hand, if they are equal, we say f is *integrable* on $[a,b]$, and in that case, the common value is *the integral* of f on $[a,b]$:

$$\int_a^b f := L\int_a^b f = U\int_a^b f.$$

Not every function is integrable, but we have the following:

Theorem. If f is a continuous function on the interval $[a,b]$, then it is integrable.

Proof. Math 112. □

Summary of essential vocabulary. Let f be a function defined on a closed interval $[a,b]$.

1. The function f is *bounded* if the set of numbers $f(x)$ for $a \leq x \leq b$ is bounded above and below, i.e., $f([a,b])$ is bounded.

2. A *partition* P of $[a,b]$ is a finite set of points

$$t_0 < t_1 < \cdots < t_n$$

   with $t_0 = a$ and $t_n = b$ and with n some positive integer. We write $P = \{t_0, t_1, \dots, t_n\}$. For the following, fix such a partition P.

3. The *subintervals* for P are the intervals

$$[t_0,t_1], [t_1,t_2], \dots, [t_{n-1},t_n].$$

   There are n subintervals in total. The i-th one is $[t_{i-1},t_i]$. Each of these intervals is contained in the interval $[a,b]$. The *length* of the i-th subinterval is $t_i - t_{i-1}$.

4. We are going to estimate the area under the graph of f with rectangles based on each subinterval. Their heights are determined by the height of the graph of f:

$$M_i := \operatorname{lub} f([t_{i-1}, t_i])$$
$$m_i := \operatorname{glb} f([t_{i-1}, t_i]).$$

If f is continuous, these numbers are the maximum and minimum values of f on the i-th subinterval.

5. The partition P determines an overestimate and an underestimate for the area under the function f. These are called the *upper and lower sums for f with respect to P*:

$$U(f,p) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) = M_1(t_1 - t_0) + M_2(t_2 - t_1) + \cdots + M_n(t_n - t_{n-1})$$
$$L(f,p) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) = m_1(t_1 - t_0) + m_2(t_2 - t_1) + \cdots + m_n(t_n - t_{n-1}).$$

The i-th summand is the area of a rectangle whose base is the i-th subinterval.

6. For each partition P that we choose, we get an upper sum (overestimate) and a lower sum (underestimate) for the area under f. In an attempt to get the best over- and underestimates, we define the *upper and lower integrals* of f on $[a, b]$:

$$U\int_a^b f := \operatorname{glb}\{U(f,P) : P \text{ a partition of } [a,b]\}$$
$$L\int_a^b f = \operatorname{lub}\{L(f,P) : P \text{ a partition of } [a,b]\}.$$

7. Finally, if the upper and lower integrals are equal (as they are when f is continuous), we define the *integral* of f to be their common value:

$$\int_a^b f := L\int_a^b f = U\int_a^b f.$$

Week 9, Wednesday: Definition of the integral. Some first examples.

We will start this class by reviewing the definition of the integral. Please see the previous lecture. The key words and notation you should know:

- Partition of a close interval $[a, b]$:

$$P = \{t_0, \dots, t_n\}$$

  with

$$a = t_0 < t_1 < \dots < t_n.$$

- The subintervals of the partition P:

$$[t_0, t_1], [t_1, t_2], \dots, [t_{n-1}, t_n].$$

  The i-th subinterval is $[t_{i-1}, t_i]$. It's length is $t_i - t_{i-1}$. You should think of each of these as a base for a rectangle.

- The y-values for f on the i-th interval

$$f([t_{i-1}, t_i]).$$

  This is the set of heights of the graph of the function sitting over the interval $[t_{i-1}, t_i]$. Think of these as the possible heights for approximating rectangles with base $[t_{i-1}, t_i]$.

-

$$M_i = \operatorname{lub} f([t_{i-1}, t_i]) \quad \text{and} \quad m_i = \operatorname{glb} f([t_{i-1}, t_i]).$$

  These are the heights for the best overestimating rectangle and underestimating rectangle, respectively.

- Upper sum and lower sum for f with respect to P:

$$U(f,p) = M_1(t_1 - t_0) + M_2(t_2 - t_1) + \cdots + M_n(t_n - t_{n-1}) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$$

$$L(f,P) = m_1(t_1 - t_0) + m_2(t_2 - t_1) + \cdots + m_n(t_n - t_{n-1}) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$$

  These are over- and underestimates for the integral.

- Upper and lower integrals:

$$U\int_a^b f := \text{glb}\,\{U(f,P) : P \text{ a partition of } [a,b]\}$$

$$L\int_a^b f := \text{lub}\,\{L(f,P) : P \text{ a partition of } [a,b]\}\,.$$

  Recall that for each partition P, we get an overestimate of the integral: $U(f,P)$. As P varies over all possible partitions, we get a whole set of overestimates. We want to take the smallest of these. The problem, it is usually the case that the set of all overestimates has no least element, just as the set $(3,8]$ has no least element. So we need to take the greatest lower bound. Similar comments apply to the lower integral.

- If $U\int_a^b f = L\int_a^b f$, the f is integrable and

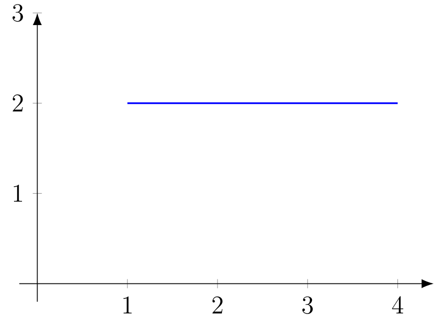
$$\int_a^b f := L\int_a^b f = U\int_a^b f.$$

  If the integral exists, the set of upper sums is usually an interval of the form $(u,v]$ for some u and v, and the set of lower sums is an interval of the form $[w,u)$ for some w. The integral is the number u that's in between the two sets but not in either. We would have

$$L\int_a^b f = \text{lub}[w,u) = u = \text{glb}(u,v] = \int_a^b f.$$

Example. It's really hard to use the definition of the integral to find the precise value of the integral. Here is one case that's easy. Suppose that f is a constant function. For example, suppose that $f(x) = 2$ for all x. We will use the definition of the integral to prove that $\int_1^4 f = 6$.

So we are considering the integral of the function $f(x) = 2$ on the interval $[1,4]$. Here is the graph:



Graph of $f(x) = 2$ on the interval $[1, 4]$.

Consider the "coarsest" partition—the one containing only the endpoints of the interval

$$P = \{1, 4\}.$$

There is only one subinterval: the interval $[1, 4]$, itself. Its length is $4 - 1 = 3$. Since $f(x) = 2$ is a constant function, we have $M_1 = m_1 = 2$. It follows that $U(f, P) = L(f, P) = 2 \cdot 3 = 6$. Since $U(f, P)$ is *an* upper sum and $U\int_1^4 f$ is a *lower bound* for upper sums, we have

$$U\int_1^4 f \leq U(f, P).$$

Similarly, since $L(f, P)$ is *a* lower sum and $L\int_1^4 f$ is an *upper bound for the lower sums*, we have

$$L(f, P) \leq L\int_1^4 .$$

Combining these inequalities, we get

$$\star\star \qquad L(f, P) \leq L\int_1^4 \leq U\int_1^4 f \leq U(f, P). \qquad \star\star$$

This string of inequalities holds not just in this example but for all functions f and for all partitions P. However, for our particular f and the partition, P, that we have chosen, we get

$$6 = L(f, P) \leq L\int_1^4 \leq U\int_1^4 f \leq U(f, P) = 6.$$

Since the number 6 appears at both ends, the upper and lower integrals must be equal to each other:

$$L\int_1^4 = U\int_1^4 f = 6.$$

This proves that

$$\int_1^4 f = 6.$$

Example. It turns out that if a function is continuous or only has a finite number of discontinuities, then the function is integrable. So a non-integrable function is necessary hard to picture. Here's an example:

$$f\colon [0,1] \to [0,1]$$
$$x \mapsto \begin{cases} 1 & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$$

Between every two distinct numbers, there is both a rational and an irrational number. So this function is in fact not continuous as any point in $[0,1]$. Let $P = \{t_0, t_1, \ldots, t_n\}$ be any partition of $[0,1]$. Consider the i-th subinterval, $[t_{i-1}, t_i]$. This interval will contain both rational and irrational numbers. Hence, $M_i = 1$ and $m_i = 0$ for all i. This means that every approximating rectangle for the upper sum will have height 1, and thus, the upper sum will be

$$\begin{aligned} U(f,P) &= 1(t_1 - t_0) + 1(t_2 - t_1) + 1(t_3 - t_2) + 1(t_4 = t_3) + \cdots + 1(t_n - t_{n-1}) \\ &= -t_0 + t_n \\ &= 0 + 1 = 1. \end{aligned}$$

Notice how all of the inner t_i cancel! You should think about this geometrically, too: what would all of the rectangles look like? The lower sum will be

$$\begin{aligned} L(f,P) &= 0(t_1 - t_0) + 0(t_2 - t_1) + 0(t_3 - t_2) + \cdots + 0(t_n - t_{n-1}) \\ &= 0. \end{aligned}$$

So *every* upper sum is 1 and *every* lower sum is 0. This means that

$$U\int_a^b f := \operatorname{glb}\{U(f,P) : P \text{ a partition of } [a,b]\} = \operatorname{glb}\{1\} = 1,$$

and

$$L\int_a^b f := \operatorname{lub}\{L(f,P) : P \text{ a partition of } [a,b]\} = \operatorname{lub}\{0\} = 0.$$

It is easy to take the glb or lub of a set that contains only one element, as in these cases. So we have

$$0 = L\int_a^b = 0 \lneq 1 = U\int_a^b f,$$

which means that f is not integrable. Now matter how we pick out partitions, the lower and upper sums are not going to get close to each other.

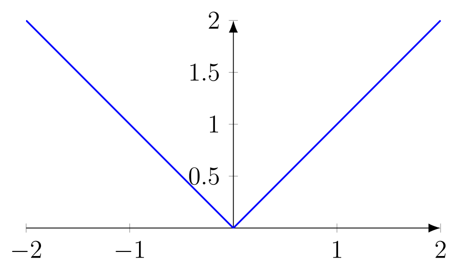
Week 9, Friday: Definition of the integral, continued.

Exercise. We'll start with a prolonged exercise designed to help get used to the definition of the integral.

1. Graph the following functions:

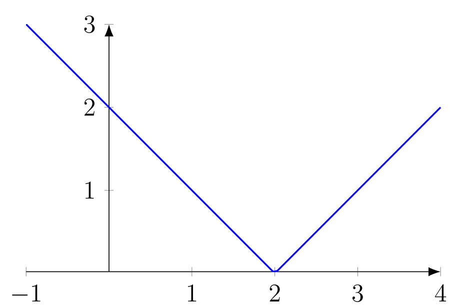
   (a)
$$\ell(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

   SOLUTION:



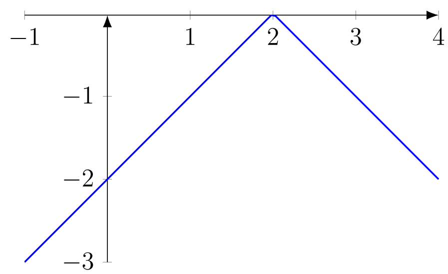
   (b) $k(x) = |x - 2|$.

   SOLUTION:



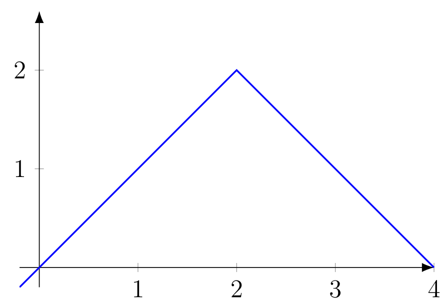
   (c) $h(x) = -|x - 2|$.

   SOLUTION:



(d) $f(x) = 2 - |x - 2|$.

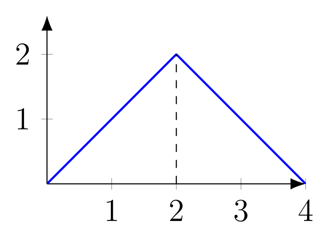
SOLUTION:



2. Consider the function $f(x) = 2 - |x - 2|$ on the interval $[0, 4]$.

   (a) Compute the area under the graph of f and above the interval $[0, 4]$ on the x-axis using high school geometry.

   SOLUTION: Divide the area into two triangles, as pictured below:

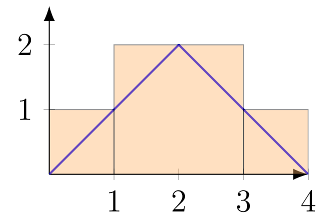


   Each triangle has base of length 2 and height 2, hence area 2. So the total area is 4.

   (b) Let $P = \{0, 1, 3, 4\}$ be a partition of $[0, 4]$. Compute $U(f, P)$ and $L(f, P)$, the upper and lower sums for f on P.

   SOLUTION: For the upper sum, we get

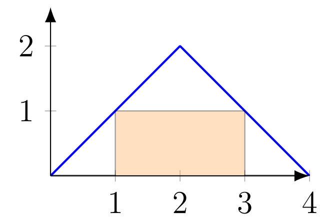
$$\begin{aligned} U(f, P) &= M_1(1 - 0) + M_2(3 - 1) + M_3(4 - 3) \\ &= 1 \cdot 1 + 2 \cdot 2 + 1 \cdot 1 \\ &= 6. \end{aligned}$$



Overestimate $U(f,P)$.

For the lower sum, we get

$$\begin{aligned}L(f,P) &= m_1(1-0)+m_2(3-1)+m_3(4-3)\\ &= 0\cdot 1+1\cdot 1+0\cdot 1\\ &= 1.\end{aligned}$$

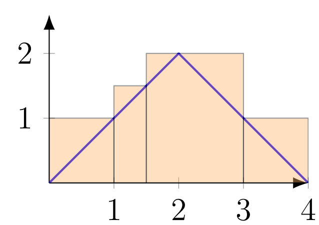


Underestimate $L(f,P)$.

(c) Now consider the partition $Q = \left\{0, 1, \frac{3}{2}, 3, 4\right\}$ of $[0,4]$. Compute $U(f,Q)$ and $L(f,Q)$. You should notice that the estimates are better.

SOLUTION: For the upper sum, we get

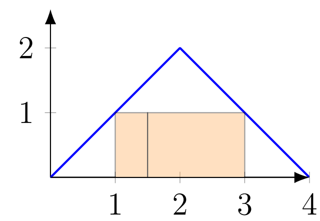
$$\begin{aligned}U(f,Q) &= M_1(1-0)+M_2(3/2-1)+M_3(3-3/2)+M_4(4-3)\\ &= 1\cdot 1+\frac{3}{2}\cdot\frac{1}{2}+2\cdot\frac{3}{2}+1\cdot 1\\ &= 5\frac{3}{4}=\frac{23}{4}.\end{aligned}$$



Overestimate $U(f,Q)$.

For the lower sum, we get

$$\begin{aligned} L(f,Q) &= m_1(1-0) + m_2(3/2-1) + m_3(3-3/2) + m_4(4-3) \\ &= 0\cdot 1 + 1\cdot\frac{1}{2} + 1\cdot\frac{3}{2} + 0\cdot 1 \\ &= 2. \end{aligned}$$

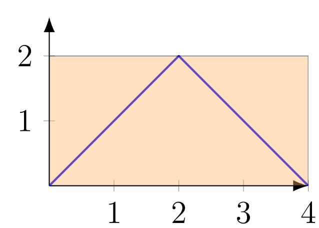


Underestimate $L(f,Q)$.

(d) Find a partition S of $[0,4]$ for which $L(f,S)$ is smallest (the worst underestimate), and find a partition S' of $[0,4]$ for which $U(f,S')$ is largest (the worst overestimate).

SOLUTION: These will be given by the partition $S = S' = \{0,4\}$ having only 2 points. For the upper sum, we get

$$U(f,S) = M_1(4-0) = 2\cdot 1 = 8.$$



Overestimate $U(f,P)$.

For the lower sum, we get

$$L(f,S) = M_1(4-0) = 0\cdot 1 = 0.$$

(e) Describe the set of real numbers that are possible upper sums as you vary the partitions. Do the same for lower sums.

SOLUTION: The set of possible upper sums is $(4,8]$, and the set of possible lower sums is $[0,4)$. The point here is that the actual area, the integral, $\int_0^4 f(x)$, is sandwiched in between. We get the value of the integral as

$$L\int_0^4 f = \text{lub}[0,4) = 4 = \text{glb}(4,8] = U\int_0^4 f.$$

(f) Consider the function

$$q(x) = \begin{cases} 2 - |x-2| & \text{if } x \neq \frac{1}{2} \\ 10 & \text{if } x = \frac{1}{2}. \end{cases}$$

What are the upper and lower sums for q on the interval $[0,4]$ with respect to the partition $P = \{0,1,3,4\}$? Compare this with the upper and lower sums for f on P.

SOLUTION: The upper sum is

$$\begin{aligned} U(q,P) &= M_1(1-0) + M_2(3-1) + M_3(4-3) \\ &= 10 \cdot 1 + 2 \cdot 2 + 1 \cdot 1 \\ &= 13, \end{aligned}$$

and the lower sum is

$$\begin{aligned} L(q,P) &= m_1(1-0) + m_2(3-1) + m_3(4-3) \\ &= 0 \cdot 1 + 1 \cdot 2 + 0 \cdot 1 \\ &= 2. \end{aligned}$$

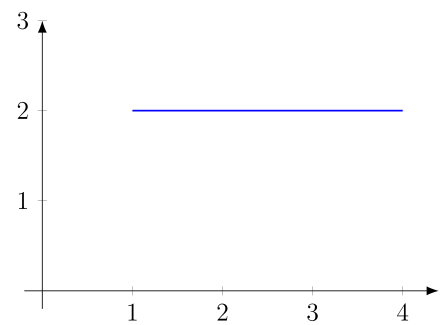
(g) What is $\int_0^4 q$?

SOLUTION: The integral does not notice an isolated discontinuity. So $\int_0^4 q = \int_0^4 f = 4$.

Week 10, Monday: The fundamental theorem of calculus.

Example. It's difficult to use the definition of the integral to find the *precise* value of the integral. One case that's easy, though, is when f is a constant function. For example, suppose that $f(x) = 2$ for all x. We will use the definition of the integral to prove that $\int_1^4 f = 6$.

So we are considering the integral of the function $f(x) = 2$ on the interval $[1, 4]$. Here is the graph:



Graph of $f(x) = 2$ on the interval $[1, 4]$.

Consider the "coarsest partition"—the one containing only the endpoints of the interval

$$P = \{1, 4\}.$$

There is only one subinterval: the interval $[1, 4]$, itself. Its length is $4 - 1 = 3$. Since $f(x) = 2$ is a constant function, we have $M_1 = m_1 = 2$. It follows that $U(f, P) = L(f, P) = 2 \cdot 3 = 6$. Since $U(f, P)$ is *an* upper sum and $U\!\int_1^4 f$ is a *lower bound* for upper sums, we have

$$U\!\int_1^4 f \leq U(f, P).$$

Similarly, since $L(f, P)$ is *a* lower sum and $L\!\int_1^4 f$ is an *upper bound for the lower* *sums*, we have

$$L(f, P) \le L\int_1^4 .$$

Combining these inequalities, we get

$$\star\star \qquad L(f, P) \le L\int_1^4 \le U\int_1^4 f \le U(f, P). \qquad \star\star$$

This string of inequalities holds not just in this example but for all functions f and for all partitions P. However, for our particular f and the partition, P, that we have chosen, we get

$$6 = L(f, P) \le L\int_1^4 \le U\int_1^4 f \le U(f, P) = 6.$$

Since the number 6 appears at both ends, the upper and lower integrals must be equal to each other:

$$L\int_1^4 = U\int_1^4 f = 6.$$

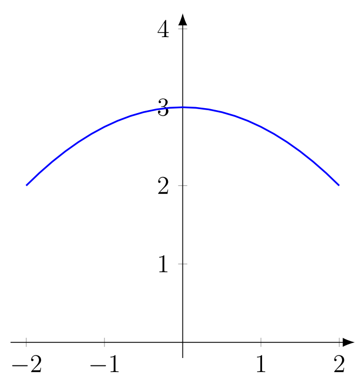
This proves that

$$\int_1^4 f = 6.$$

The fundamental theorem of calculus. In homework, we considered the function

$$f(x) = -\frac{1}{4}x^2 + 3$$

on the interval $[-2, 2]$:



Graph of $f(x) = -\frac{1}{4}x^2 + 3$.

To estimate the integral, $\int_{-2}^{2} f$, we took the partition

$$Q = \left\{-2, -\frac{1}{4}, 1, 2\right\}$$

and found

$$L(f, Q) = \frac{143}{16} \approx 8.9 < 11.7 \approx \frac{3001}{256} = U(f, Q).$$

The actual value of the integral is $\int_{-2}^{2} f = 32/3 \approx 10.7$. Averaging the upper and lower sums gives an OK approximation (given how few points are in the partition).

It is possible compute the *precise* value of the integral, $\int_{-2}^{2} f$, straight from the definition of the integral, but rather painful. To outline what needs to be done, imagine dividing the interval $[-2, 2]$ into n parts, each part with length $4/n$. That gives a partition P_n for each $n = 1, 2, 3, \ldots$. (We did something similar earlier for the function $g(x) = 2x$). Using somewhat complicated algebra, it is possible to find closed formulas for the lower and upper sums, and we will get

$$L(f, P_n) < L \int_{-2}^{2} f \leq U \int_{-2}^{2} f < U(f, P_n).$$

From the formulas for the sums, we would then compute limits and find

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n).$$

which implies the upper and lower *integrals* are both equal to this limit, and thus, we have computed the integral.

There is a much easier way to compute the integral, though. It comes from the following magic known as the fundamental theorem of calculus:

Fundamental theorem of calculus (FTC). Let f be an integrable function on the interval $[a, b]$. Suppose there is a function g which is continuous on $[a, b]$ and differentiable on (a, b) such that $g' = f$. Then

$$\int_a^b f = g(b) - g(a).$$

In other words,

$$\int_a^b g' = g(b) - g(a).$$

To make the name of the variable explicit, you could write the FTC as follows:

$$\int_a^b f(x)\, dx = g(b) - g(a).$$

Definition. An *antiderivative* of a function f is any function g such that $g' = f$.

So the FTC allows us to compute the integral of a function on an interval $[a, b]$ by simply computing the *net change in the function's antiderivative over the interval.*

Example. Consider the earlier example

$$f(x) = -\frac{1}{4}x^2 + 3$$

on the interval $[-2, 2]$. To integrate f using the FTC, we first find a function g whose derivative is f. We can take

$$g(x) = -\frac{1}{12}x^3 + 3x.$$

(Every other possibility differs from the g we have chosen by the addition of a constant, and that does not affect the answer.) The FTC then says

$$\begin{aligned}
\int_{-2}^{2} f &= g(2) - g(-2) \\
&= \left(-\frac{1}{12}(2)^3 + 3(2)\right) - \left(-\frac{1}{12}(-2)^3 + 3(-2)\right) \\
&= \left(-\frac{2}{3} + 6\right) - \left(\frac{2}{3} - 6\right) \\
&= -\frac{4}{3} + 12 \\
&= \frac{32}{3}.
\end{aligned}$$

Notation. We rewrite the previous calculation using some convenient notation:

$$\begin{aligned}
\int_{-2}^{2} \left(-\frac{1}{4}x^3 + 4\right) dx &= \left(-\frac{1}{12}(x)^3 + 3(x)\right)\Bigg|_{-2}^{2} \\
&= \left(-\frac{1}{12}(2)^3 + 3(2)\right) - \left(-\frac{1}{12}(-2)^3 + 3(-2)\right) \\
&= \frac{32}{3}.
\end{aligned}$$

The big vertical line at the end of the first line of the calculation indicates that we should evaluate the function preceding the vertical line at the two endpoints, 2 and -2, then subtract.

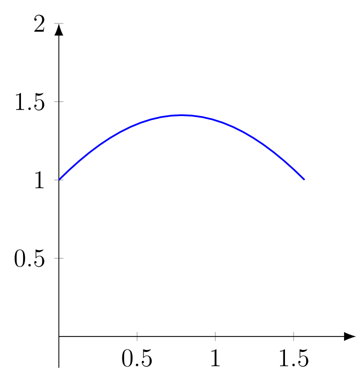
Example. Here we use the fundamental theorem to compute the area under $f(x) = 2$ on $[1, 4]$, the example we computed at the beginning of this lecture:

$$\int_1^3 2\,dx = 2x\Big|_1^4 = 2 \cdot 4 - 2 \cdot 1 = 6.$$

Practice. Compute the following integrals:

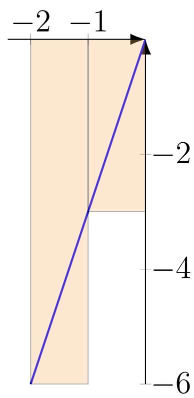
1. $f(x) = x^3$ on $[0, 2]$.
2. $f(x) = x^3$ on $[1, 2]$.
3. $f(x) = \cos(x)$ on $[0, \pi/2]$.
4. $f(x) = \cos(x) + 1$ on $[0, \pi/2]$.
5. $f(x) = \cos(x) + \sin(x)$ on $[0, \pi/2]$.

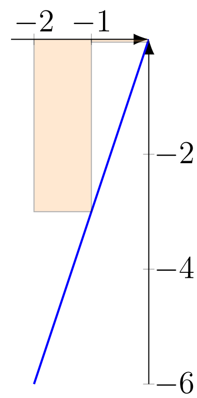
Here is a graph of the last function in the above list:



Graph of $f(x) = x^2$ and its tangent line at $x = 3$.

Area below the x-axis is counted negatively. We have been referring to the value computed by the integral as the *area* under the graph of a function. It turns out, though, that if the graph is below the x-axis, then the area *above* the graph and below the x-axis is counted negatively. For example, consider the function $f(x) = 3x$ on the interval $[-2, 0]$. Compute the upper and lower sums for f with respect to the partition $[-2, 1, 0]$:



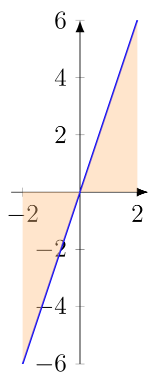


$$L(f,P) = -6 - 3 = -9 \qquad U(f,P) = -3 - 0 = -3.$$

The actual area between the graph and the x-axis is 6, but the integral will count this as -6:

$$\int_{-2}^{0} 3x\,dx = \frac{3}{2}x^2\bigg|_{-2}^{0} = \frac{3}{2}\cdot 0^2 - \frac{3}{2}\cdot -2^2 = -6.$$

What about integrating $f(x) = 3x$ on $[-2, 2]$?



Graph of $f(x) = 3x$.

The area above the axis is counted positively and the area below the axis is counted negatively, so they cancel in the integral:

$$\int_{-2}^{2} 3x\,dx = \frac{3}{2}x^2\bigg|_{-2}^{2} = \frac{3}{2}\cdot 2^2 - \frac{3}{2}\cdot -2^2 = 0.$$

If you want the actual area between the graph of f and the x-axis, you would need to break the integral into two parts—one for the area above the axis, and one for the area below the graph—and subtract:

$$\int_0^2 3x\,dx - \int_{-2}^0 3x\,dx = 6 - (-6) = 12.$$

Week 10, Wednesday: Properties of the integral. Integration practice.

Properties of integrals.

- Suppose that f and g are integrable on $[a, b]$ and $c \in \mathbb{R}$. then
$$\int_a^b (f+g) = \int_a^b f + \int_a^b g \quad \text{and} \quad \int_a^b cf = c\int_a^b f.$$
- Suppose that f and g are integrable on $[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$. Then $\int_a^b f \leq \int_z^b g$. In other words, integration preserves inequalities.
- Suppose f is integrable on $[a, b]$ and $a < c < b$. Then
$$\int_a^b f = \int_a^c f + \int_c^b f.$$
- If f is continuous on $[a, b]$, then $\int_a^b f$ exists. In general,
$$\text{differentiable} \quad \Longrightarrow \quad \text{continuous} \quad \Longrightarrow \quad \text{integrable}.$$

Proof. Math 112 (maybe). □

Practice with the FTC. Recall the FTC: Roughly, if f is integrable and $g' = f$, then
$$\int_a^b f = g(x)\big|_a^b := g(b) - g(a).$$

For example,
$$\begin{aligned}\int_1^2 (3x^2 + 4x - 5) &= (x^3 + 2x^2 - 5x)\big|_1^2 \\ &= (2^3 + 2\cdot 2^2 - 5\cdot 2) - (1^3 + 2\cdot 1^2 - 5\cdot 1) \\ &= 6 - (-2) = 8.\end{aligned}$$

The properties of the integral introduced above allow you to break up the integral into simpler pieces:

$$\int_1^2 (3x^2 + 4x - 5) = 3\int_1^2 x^2 + 4\int_1^2 x - 5\int_1^2 1 = \text{ etc.}$$

Try the following problems:

1. $\int_0^3 (2e^x + 5x)\, dx$.
2. $\int_0^\pi \sin(x)\, dx$.
3. $\int_0^2 (3x^2 + 2x + 1)(x^3 + x^2 + x)^{99}\, dx$.

Antiderivatives. The key to using the FTC to compute an integral $\int_a^b f$ is to find a function g such that $g' = f$. Such a function g is called an *antiderivative* of f.

Example. The function $x^3/3$ is an antiderivative of $f(x) = x^2$. The most general antiderivative of f is $x^3/3 + c$ where is c is any constant. We use the following notation:

$$\int x^2 = \frac{x^3}{3} + c$$

or

$$\int x^2\, dx = \frac{x^3}{3} + c.$$

Compare the above with

$$\int_0^1 x^2\, dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}.$$

The integral $\int_0^1 x^2\, dx$ is what we have defined using lower and upper sums. It is sometimes called the *definite* integral to distinguish it from the indefinite integral $\int x^2\, dx$, which is just a general antiderivative which can be used to evaluate the definite integral via the FTC.

Compute the following indefinite integrals:

1. $\int x^n\, dx$ for $n = 1, 2, 3, \ldots$
2. $\int x^{1/2}\, dx$.
3. $\int x^{3/2}\, dx$.

4. $\int 5x^7 + 2x^3 + 4\, dx$.

5. $\int \cos(x) + 3e^x\, dx$.

6. $\int e^{ax}$ where a is a nonzero constant.

7. $\int \cos(4x)\, dx$.

8. $\int x(3x^2 + 5)^{100}\, dx$.

9. $\int x^2 e^{x^3}\, dx$.

Week 10, Friday: Integration by substitution. Integration by parts.

Finding antiderivatives mean reversing the operation of taking derivatives. Today we'll consider reversing the chain rule and the product rule.

Substitution technique. Recall the chain rule:

$$(f(g(x)))' = f'(g(x))g'(x).$$

In terms of antiderivatives, this means

$$\int f'(g(x))g'(x)\,dx = f(g(x)) + c.$$

For example,

$$\int 10(x^3+4x+2)^9(3x^2+4)\,dx = (x^3+4x+2)^{10} + c.$$

Here, $f(x) = x^{10}$ and $g(x) = x^3 + 4x + 2$.

The technique of substitution is a formalism that helps in detecting the presence of the chain rule. Here's how it works. We know that

$$\int f'(g(x))g'(x)\,dx = f(g(x)) + c.$$

Define $u(x) = g(x)$. Then

$$\frac{du}{dx} = g'(x).$$

We abuse this notation by writing

$$du = g'(x)dx$$

and then substitute into the integral to get

$$\int f'(g(x))g'(x)\,dx = \int f'(u)du.$$

Then, by the FTC, we get

$$\int f'(u)\,du = f(u) + c.$$

Substituting back, using $u = g(x)$, we get

$$\int f'(g(x))g'(x)\,dx = \int f'(u)\,du = f(u) + c = f(g(x)) + c.$$

Example. Consider the indefinite integral

$$\int 3x^2(x^3+5)^6\,dx.$$

You may be able to immediately see how the chain rule applies. If not, as a general rule of thumb, look for a part of the integrand (the function you're integrating) that is "inside" another function and substitute. In this case, an obvious choice is to let

$$u = x^3 + 5.$$

Then using our notation from above,

$$du = 3x^2\,dx.$$

Substitute and integrate:

$$\int 3x^2(x^3+5)^6\,dx = \int u^6\,du = \frac{1}{7}u^7 + c.$$

To get the final solution, substitute back:

$$\int 3x^2(x^3+5)^6\,dx = \frac{1}{7}(x^3+5)^7 + c.$$

Example. Integrate $\int x^2\cos(x^3)\,dx$. The "inside" function here is $u = x^3$. We get

$$du = 3x^2\,dx.$$

Therefore,

$$x^2\,dx = \frac{1}{3}du.$$

Now substitute:

$$\begin{aligned}\int x^2 \cos(x^3)\, dx &= \int \frac{1}{3} \cos(u)\, du \\ &= \frac{1}{3} \sin(u) + c \\ &= \frac{1}{3} \sin(x^3) + c.\end{aligned}$$

Example. Here is a trickier example:

$$\int x\sqrt{1+5x}\, dx.$$

The inside function is $u = 1 + 5x$. So

$$du = 5\, dx \quad \Rightarrow \quad dx = \frac{1}{5}\, du.$$

We now need to substitute into the original integral to obtain an integral solely in the variable u—we need to get rid of all of the xs. Making a partial substitution in $x\sqrt{1+5x}\, dx$, we would get

$$x\sqrt{1+5x}\, dx = \frac{1}{5}\, x\sqrt{u}\, du,$$

but we need to get rid of the x remaining in this expression. Here's how: since $u = 1 + 5x$, we can solve for x in terms of u:

$$u = 1 + 5x \quad \Rightarrow \quad x = \frac{1}{5}(u-1).$$

Thus,

$$x\sqrt{1+5x}\, dx = \frac{1}{5}\, x\sqrt{u}\, du = \frac{1}{25}(u-1)\sqrt{u}\, du.$$

So

$$\begin{aligned}
\int x\sqrt{1+5x}\,dx &= \int \frac{1}{25}(u-1)\sqrt{u}\,du \\
&= \int \frac{1}{25}(u-1)u^{1/2}\,du \\
&= \frac{1}{25}\int (u-1)u^{1/2}\,du \\
&= \frac{1}{25}\int (u^{3/2}-u^{1/2})\,du \\
&= \frac{1}{25}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)+c \\
&= \frac{1}{25}\left(\frac{2}{5}(1+5x)^{5/2}-\frac{2}{3}(1+5x)^{3/2}\right)+c.
\end{aligned}$$

WARNING. Be careful with limits of integration when using substitutions. For example, using the substitution $u = x^5 + 1$ and $du = 5x^4\,dx$, we get

$$\int_0^1 x^4(x^5+1)^6\,dx = \frac{1}{5}\int_1^2 u^6\,du = \frac{1}{35}u^7\Big|_1^2 = \frac{1}{35}(2^7-1^7) = \frac{127}{35}.$$

The limits of integration change after the substitute since $u = 1$ when $x = 0$ and $u = 2$ when $x = 1$.

As an alternative, you could first just compute the *indefinite* integral (using the same substitution:

$$\int (x^5+1)^6\,dx = \frac{1}{5}\int u^6\,du = \frac{1}{35}u^7 = \frac{1}{35}(x^5+1)^7 + c.$$

Then use the FTC:

$$\int_0^1 x^4(x^5+1)^6\,dx = \frac{1}{35}(x^5+1)^7\Big|_0^1 = \frac{127}{35}.$$

Integration by parts. The integration technique called *integration by parts* originates from the product rule:

$$(uv)' = u'v + ug'.$$

Integrate:

$$\int (uv)' = \int (u'v+uv') = \int u'v + \int uv'.$$

Now, $\int (uv)'$ is the indefinite integral; so it must be a function whose derivative is $(uv)'$, but that's easy: uv. So

$$uv = \int u'v + \int uv'.$$

We can use a more notation to specify the argument of the function (the independent variable):

$$u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx.$$

Using the notation $du = u'(x)dx$ and $dv = v'(x)dx$, we can write

$$uv = \int v\,du + \int u\,dv.$$

Rearranging, we get the form that is useful for integration:

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

The utility of this formula is that it might be that $\int v\,du$ is an easier integral than $\int v\,du$.

Example. Compute $\int xe^x\,dx$? Note that since $(e^x)' = e^x$, it's trivial to integrate e^x: we have $\int e^x\,dx = e^x + c$. To integrate xe^x by parts, we need to choose u and dv appropriately. The following choice works:

$$\begin{aligned} u &= e^x \\ dv &= x\,dx. \end{aligned}$$

We then need to find du and v:

$$\begin{aligned} u &= x & du &= dx \\ dv &= e^x\,dx & v &= e^x. \end{aligned}$$

Applying the boxed formula, above:

$$\begin{aligned} \int xe^x\,dx &= \int u\,dv \\ &= uv - \int v\,du \\ &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + c. \end{aligned}$$

It is easy to check that the solution is correct: differentiate $xe^x - e^x + c$, and you will get xe^x. (You'll need the product rule, naturally.)

We can then use the antiderivative we've found to compute definite integrals. For example,

$$\begin{aligned}\int_0^1 xe^x\,dx &= (xe^x - e^x)\Big|_0^1 \\ &= (1 \cdot e^1 - e^1) - (0 \cdot e^0 - e^0) \\ &= (e - e) - (0 - 1) \\ &= 1.\end{aligned}$$

Example. Compute $\int x\cos(x)\,dx$. By parts:

$$\begin{aligned} u &= x & du &= dx \\ dv &= \cos(x)\,dx & v &= \sin(x).\end{aligned}$$

Then

$$\begin{aligned}\int x\cos(x)\,dx &= \int u\,dv \\ &= uv - \int v\,du \\ &= x\sin(x) - \int \sin(x)\,dx \\ &= x\sin(x) + \cos(x) + c.\end{aligned}$$

Check:

$$\begin{aligned}(x\sin(x) + \cos(x) + c)' &= (x\sin(x))' + \cos'(x) \\ &= (\sin(x) + x\cos(x)) - \sin(x) \\ &= x\cos(x).\end{aligned}$$

Challenge. Compute $\int e^x\cos(x)\,dx$ by parts.

Week 11, Monday: Finish integration by parts. The logarithm.

Another integration by parts example. Recall the formula for integration by parts:

$$\int u\,dv = uv - \int v\,du.$$

Let's apply this to compute the indefinite integral

$$\int e^x \cos(x)\,dx.$$

Let

$$\begin{aligned} u &= e^x & du &= e^x\,dx \\ dv &= \cos(x)\,dx & v &= \sin(x). \end{aligned}$$

We get

$$\int e^x \cos(x)\,dx = e^x \sin(x) - \int e^x \sin(x)\,dx. \tag{29.1}$$

So we now need to compute $\int e^x \sin(x)\,dx$. Try again by parts, this time with

$$\begin{aligned} u &= e^x & du &= e^x\,dx \\ dv &= \sin(x)\,dx & v &= -\cos(x). \end{aligned}$$

We get

$$\int e^x \sin(x)\,dx = -e^x \cos(x) + \int e^x \cos(x)\,dx.$$

It may look like we've gone in circles. But if you carefully substitute what we've just calculated back into equation (29.1), we get

$$\begin{aligned} \int e^x \cos(x)\,dx &= e^x \sin(x) - \int e^x \sin(x)\,dx \\ &= e^x \sin(x) - (-e^x \cos(x) + \int e^x \cos(x)\,dx) \\ &= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx). \end{aligned}$$

Add $\int e^x \cos(x)\,dx$ to both sides to get

$$2\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x).$$

Therefore, the final solution is

$$\int e^x \cos(x)\,dx = \frac{1}{2}(\sin(x) + \cos(x))e^x + c.$$

(It's easy to take the derivative of the right-hand side, using the product rule, to check that we've got the right answer.)

The logarithm function. To define the logarithm function, we first need to talk about a theorem sometimes called the second fundamental theorem of calculus. It says that every continuous function has an antiderivative.

Theorem. (FTC2) Suppose f is a continuous function on an open interval I containing a point a. Define

$$g(x) = \int_a^x f(t)\,dt$$

for each $x \in I$. Then $g'(x) = f(x)$ for each $x \in I$.

Proof. Math 112. □

Example. Let $f(x) = x^5$ and define

$$g(x) = \int_0^x f(t)\,dt = \int_0^x t^5\,dt.$$

Therefore,

$$\begin{aligned} g(x) &= \int_0^x t^5\,dt \\ &= \frac{1}{6}t^6\Big|_{t=0}^{x} \\ &= \frac{1}{6}x^6, \end{aligned}$$

and we have $g'(x) = f(x)$, as claimed. (Exercise: check that if we let $g(x) = \int_a^x x^5\,dt$, for any constant a, we'd still get $g'(x) = f(x)$.)

Example. Here is an example of an integral that does not have a nice closed form:

$$b(x) = \int_0^x e^{-t^2+\cos(t)}\,dt.$$

However, by FTC2, we know its derivative

$$b'(x) = e^{-x^2+\cos(x)}.$$

The logarithm function. Recall that for every real number α, we have $(x^\alpha)' = \alpha x^{\alpha-1}$. Therefore,

$$\int x^\alpha \, dx = \frac{1}{\alpha+1} x^{\alpha+1} + c$$

as long as $\alpha \neq 1$. So the case $\alpha = -1$ is somewhat mysterious:

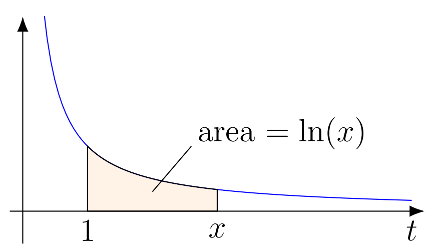
$$\int x^{-1} \, dx = \int \frac{1}{x} \, dx.$$

We handle this case by making up a name for it—the logarithm:

Definition. For $x > 0$, the natural logarithm is

$$\ln(x) = \int_1^x \frac{1}{t} \, dt.$$

So the natural logarithm is, by definition, the area under the graph of $f(t) = 1/t$ from $t = 1$ to $t = x$:



Graph of $f(t) = \frac{1}{t}$.

By FTC2, we have

$$(\ln(x))' = \frac{1}{x}.$$

Week 11, Wednesday: More logarithms.

Recall the definition of the natural logarithm from last time:

Definition. For $x > 0$, the natural logarithm is

$$\ln(x) = \int_1^x \frac{1}{t}\, dt.$$

Properties of the logarithm.

1. $(\ln(x))' = \dfrac{1}{x}$.

   Proof. This follows directly from FTC2:

$$(\ln(x))' = \left(\int_1^x \frac{1}{t}\, dt\right)' \overset{\text{FTC2}}{=} \frac{1}{x}.$$

□

2. $\ln(x)$ is an increasing function (i.e., it has positive slope) and its graph is concave down.

   Proof. It's increasing since its derivative is $1/x$, which is positive for $x > 0$. (Recall that we have only defined $\ln(x)$ for positive x.) It's concave down since

$$(\ln(x))'' = \left(\frac{1}{x}\right)' = -\frac{1}{x^2} < 0.$$

□

3.

$$\ln(x) \begin{cases} < 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ > 0 & \text{for } x > 1. \end{cases}$$

Proof. We have, so far, only defined the integral $\int_a^b g$ of a function g on an interval $[a, b]$ with $a < b$ using limits of upper and lower sums. The definitions when $a = b$ or $a > b$ are as follows:

Definition.

(a) $\displaystyle\int_a^a g(x)\,dx = 0.$

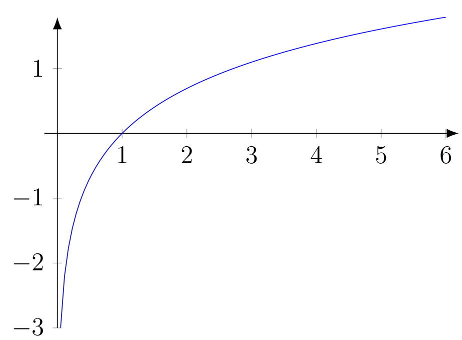
(b) If $a > b$, then $\int_a^b g(x)\,dx = -\int_b^a g(x)\,dx$.

The result about $\ln(x)$ now easily follows. In particular, if $0 < x < 1$, we have

$$\ln(x) = \int_1^x \frac{1}{t}\,dt \overset{\text{def}}{=} -\int_x^1 \frac{1}{t}\,dt < 0,$$

Note that $\int_x^1 1/t\,dt > 0$ for $x < 1$ since it's the area under the graph of $1/t$ from $t = x$ to $t = 1$ and $1/t > 0$ for $t > 0$. Similarly, if $x > 1$, then $\ln(x) > 0$ since it's the area under the graph of a positive function. □

4. The graph of $\ln(x)$ looks something like this:



Graph of $\ln(x)$.

5. For $x > 0$ and $y > 0$, we have

$$\ln(xy) = \ln(x) + \ln(y).$$

Proof. Homework. □

6. $\ln(x^n) = n\ln(x)$ for $n = 0, 1, 2, \ldots$

Proof. The cases $n = 0$ and $n = 1$ are immediate:

$$\ln(x^0) = \ln(1) = 0 = 0 \cdot \ln(x)$$
$$\ln(x^1) = \ln(x) = 1 \cdot \ln(x).$$

For $n = 2$, we use the fact that $\ln(xy) = \ln(x) + \ln(y)$ for all $x, y > 0$, and we set $x = y$ to get

$$\ln(x^2) = \ln(x \cdot x) = \ln(x) + \ln(x) = 2\ln(x).$$

For the case $n = 3$, we use the case $n = 2$ that we just established:

$$\ln(x^3) = \ln(x \cdot x^2) = \ln(x) + \ln(x^2) = \ln(x) + 2\ln(x) = 3\ln(x).$$

Similarly,

$$\ln(x^4) = \ln(x \cdot x^3) = \ln(x) + \ln(x^3) = \ln(x) + 3\ln(x) = 4\ln(x),$$

and so on. Formally, one would prove the complete result using induction. □

7. $\ln(x^{-n}) = -n\ln(x)$ for $n = 1, 2, \ldots$

   Proof. Let n be a positive integer. We have $x^n \cdot x^{-n} = 1$. Therefore,

   $$0 = \ln(1) = \ln(x^n \cdot x^{-n}) = \ln(x^n) + \ln(x^{-n}) = n\ln(x) + \ln(x^{-n}).$$

   So $n\ln(x) + \ln(x^{-n}) = 0$, and the result follows. □

8. For every real number α, we have $\ln(x^\alpha) = \alpha\ln(x)$.

   Proof. We have proved the result for α any integer. For arbitrary real numbers, see Math 112. The problem is knowing the definition of x^α. □

9. $\ln\left(\dfrac{x}{y}\right) = \ln(x) - \ln(y)$.

   Proof. Homework. □

10. $\lim_{x\to\infty} \ln(x) = \infty$.

    Proof. As a special case of $ln(x^n) = n\ln(x)$, take $x = 2$. We get, $\ln(2^n) = n\ln(2)$. We know that $\ln(2) > 0$, hence,

    $$\lim_{n\to\infty} \ln(2^n) = n\ln(2) = \infty.$$

    □

Examples.

-

$$\begin{aligned}\left(\ln(\sqrt{x^3+2x})\right)' &= \left(\ln((x^3+2x)^{1/2})\right)' \\ &= \frac{1}{(x^3+2x)^{1/2}} \cdot ((x^3+2x)^{1/2})' \\ &= \frac{1}{(x^3+2x)^{1/2}} \cdot \left(\frac{1}{2}(x^3+2x)^{-1/2}(3x^2+2)\right) \\ &= \frac{3x+2}{2(x^3+2x)}.\end{aligned}$$

  Easier alternative:

$$\begin{aligned}\left(\ln(\sqrt{x^3+2x})\right)' &= \left(\ln((x^3+2x)^{1/2})\right)' \\ &= \left(\frac{1}{2}\ln(x^3+2x)\right)' \\ &= \frac{1}{2}\left(\ln(x^3+2x)\right)' \\ &= \frac{1}{2} \cdot \frac{3x+2}{x^3+2x}.\end{aligned}$$

- $\displaystyle\int \frac{x}{x^2+4}\,dx$. Let $u = x^2+4$. Then $u = 2x\,dx$. Then,

$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(u) + c = \frac{1}{2}\ln(x^2+4) + c.$$

- $\int \tan\theta\,d\theta = \int \frac{\sin(\theta)}{\cos(\theta)}\,d\theta$. Let $u = \cos(\theta)$. Then $du = -\sin(\theta)\,d\theta$. So

$$\begin{aligned}\int \tan\theta\,d\theta &= \int \frac{\sin(\theta)}{\cos(\theta)}\,d\theta \\ &= -\int \frac{du}{u} \\ &= -\ln(u) + c \\ &= -\ln(\cos(\theta)) + c \\ &= \ln((\cos(\theta))^{-1}) + c \\ &= \ln(\sec(\theta)) + c.\end{aligned}$$

Therefore,

$$\int \tan\theta \, d\theta = \ln(\sec(\theta)) + c.$$

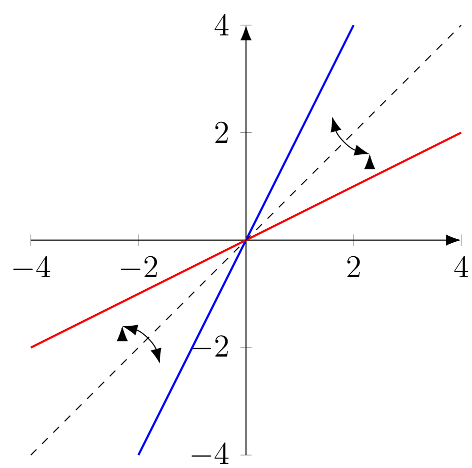
Week 11, Friday: The inverse function theorem and the exponential function.

The inverse function theorem and exponentials

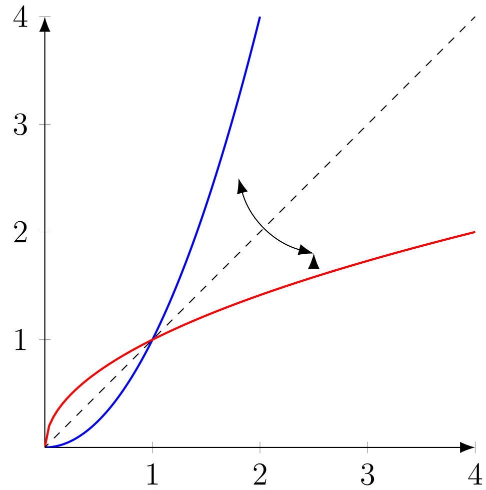
Definition. Functions f and g are inverses of each other if

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x.$$

Examples.



Graphs of inverse functions $f(x) = 2x$ and $g(x) = \frac{1}{2}x$.

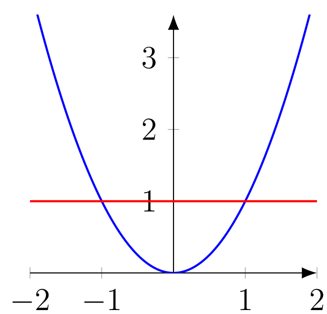


Graphs of inverse functions $f(x) = x^2$ and $g(x) = \sqrt{x}$.

Definition. A function f is one-to-one if $x \neq y$ implies $f(x) \neq f(y)$.

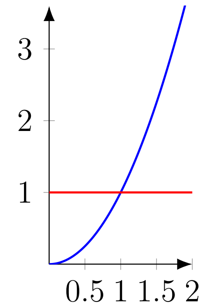
So f is one-to-one if it does not send two points to the same point. Graphically, this means that no horizontal line will meet the graph of f more than once.

Example. The function $f(x) = x^2$ as a function of the whole real number line is not one-to-one. For instance $f(1) = f(-1)$. Graphically, there exist horizontal lines meeting the graph of f in more than one point:



$f(x) = x^2$ fails the horizontal line test on $(-\infty, \infty)$.

However, if we restrict $f(x) = x^2$ to be a function on $[0, \infty)$, it is one-to-one:



$f(x) = x^2$ passes the horizontal line test on $[0, \infty)$.

Proposition. If the function f is one-to-one, it has an inverse.

Example. Considering $f(x) = x^2$ as a function on $[0, \infty)$, then it has an inverse: $g(x) = \sqrt{x}$.

Theorem. (Inverse function theorem, (IFT).) Suppose f is differentiable, and suppose f has an inverse g. Then g is differentiable and

$$g'(x) = \frac{1}{f'(g(x)}$$

provided $f'(g(x)) \neq 0$.

Proof. We can give a proof of part of this theorem. Suppose g is differentiable. Since f and g are inverses, we have $f(g(x)) = x$. Take derivatives and apply the chain rule:

$$1 = (x)' = (f(g(x))' = f'(g(x))g'(x).$$

So $1 = f'(g(x))g'(x)$. Solve for $g'(x)$ to get

$$g'(x) = \frac{1}{f'(g(x))}.$$

□

Example. Let's check the IFT with an example. The functions $f(x) = x^2$ and $g(x) = \sqrt{x}$ are inverse functions on $[0, \infty)$. We have

$$g'(x) = (x^{1/2})' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

Let's compare this with $1/f'(g(x))$. We have $f'(x) = 2x$, and $g(x) = \sqrt{x}$. So

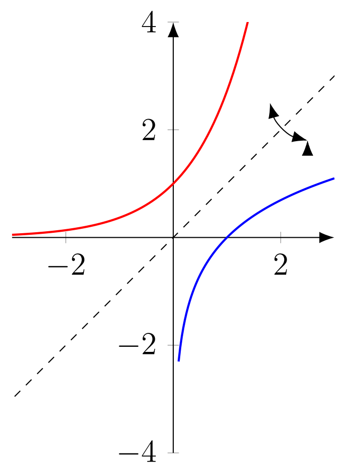
$$\frac{1}{f'(g(x))} = \frac{1}{f'(\sqrt{x})} = \frac{1}{2\sqrt{x}}.$$

The exponential function. Recall that $\ln(x)' = 1/x > 0$ for all $x > 0$. Thus, $\ln(x)$ is always increasing. In particular, this means that $\ln(x)$ has an inverse function. By definition, the exponential function,

$$\exp(x)$$

is the inverse of $\ln(x)$. In other words,

$$\exp(\ln(x)) = \ln(\exp(x)) = x.$$



Graphs of inverse functions $\ln(x)$ and $\exp(x)$.

Properties of the exponential function.

1. $\exp(0) = 1$.

   Proof. Since $\ln(1) = 0$ and $\exp(\ln(x)) = x$ for all $x > 0$, we have

$$\exp(0) = \exp(\ln(1)) = 1.$$

□

2. $\exp(x + y) = \exp(x)\exp(y)$.

   Proof. Recall that $\ln(xy) = \ln(x) + \ln(y)$ for all $x, y > 0$. Therefore,

$$\ln(\exp(x)\exp(y)) = \ln(\exp(x)) + \ln(\exp(y)) = x + y.$$

So $\ln(\exp(x)\exp(y)) = x + y$. Apply the exp function to both sides and use the fact that it is inverse to ln:

$$\begin{aligned}\ln(\exp(x)\exp(y)) = x + y \quad &\Longrightarrow \quad \exp(\ln(\exp(x)\exp(y))) = \exp(x+y) \\ &\Longrightarrow \quad \exp(x)\exp(y) = \exp(x+y).\end{aligned}$$

□

3. $\exp'(x) = \exp(x)$.

   Proof. This follows from the inverse function theorem, but we can see it directly from the chain rule:

$$\begin{aligned}\ln(\exp(x)) = x \quad &\Longrightarrow \quad (\ln(\exp(x)))' = (x)' \\ &\Longrightarrow \quad \frac{1}{\exp(x)}\exp'(x) = 1 \\ &\Longrightarrow \quad \exp'(x) = \exp(x).\end{aligned}$$

□

The number e and exponentiation. We define the number e, Euler's constant, as follows:

$$e := \exp(1).$$

We would like to show that from this one simple definition it follows that

$$e^x = \exp(x) \tag{31.1}$$

for *all* real numbers x. The problem arises is understanding what is meant by taking a number to a power. We break this up into cases:

Case 1. For the exponent $n = 0$, we take $e^0 = 1$, by definition. We have already seen that since $\ln(1) = 0$, we have $\exp(0) = 1$. So in that case, equation (31.1) holds: $e^0 = \exp(0)$.

Case 2. For the exponent $n = 1$, we have $e^1 = e$, by definition of exponentiation, and we have $e = \exp(1)$, by definition of e. So the equation holds here, too.

Case 3. Suppose $n = 2, 3, \ldots$ Here, we repeatedly use the fact we saw earlier: since the logarithm converts products to sums, $\ln(xy) = \ln(x) + \ln(y)$, it follows that the exponential function converts sums to products, $\exp(x+y) = \exp(x)\exp(y)$. It follows that

$$\begin{aligned}&e^2 = e \cdot e = \exp(1)\exp(1) = \exp(1+1) = \exp(2) \\ &e^3 = e \cdot e \cdot e = \exp(1)\exp(1)\exp(1) = \exp(1+1+1) = \exp(3) \\ &e^4 = e \cdot e \cdot e \cdot e = \exp(1)\exp(1)\exp(1)\exp(1) = \exp(1+1+1+1) = \exp(4),\end{aligned}$$

and so on.

Case 4. What about negative exponents? For $n = 1, 2, 3, \ldots$, we would like to show that $e^{-n} = \exp(-n)$. First: what does e^{-n} mean? Answer: by definition

$$e^{-n} = 1/e^n.$$

Substituting in the definition of e, then, what we need to show is that

$$\exp(-n) = \frac{1}{e^n} = \frac{1}{\exp(n)}.$$

(The second equality above follows since we have already established that $e^n = \exp(n)$ for $n = 0, 1, 2, \ldots$) Here is a nice argument to establish that fact (recalling that $\exp(0) = 1$):

$$1 = \exp(0) = \exp(n - n) = \exp(n + (-n)) = \exp(n)\exp(-n).$$

So $1 = \exp(n)\exp(-n)$, and the result follows.

Case 5. What about rational exponents? Consider the fraction a/b where a and b are integers. By definition, $e^{a/b}$ is the number such that

$$(e^{a/b})^b = e^a.$$

For instance, multiplying $e^{1/2}$ by itself gives e. For a warm-up, we will show that $e^{1/2} = \exp(1/2)$. This just means that we need to show that multiplying $\exp(1/2)$ by itself should give e. We get that from the following calculation (again involving the formula $\exp(x + y) = \exp(x)\exp(y)$):

$$\exp(1/2)\exp(1/2) = \exp(1/2 + 1/2) = \exp(1) = e.$$

What about $e^{2/5}$, the number which when multiplied by itself 5 times gives e^2. Here we have

$$\begin{aligned} \exp(2/5)\exp(2/5)\exp(2/5)\exp(2/5)\exp(2/5) &= \exp(2/5 + 2/5 + 2/5 + 2/5 + 2/5) \\ &= \exp(2) \\ &= e^2. \end{aligned}$$

The last step follows from Case 3, above. We have just shown that $\exp(2/5) = e^{2/5}$. In general, for an arbitrary fraction a/b, we have

$$\begin{aligned} \underbrace{\exp(a/b)\exp(a/b)\cdots\exp(a/b)}_{b \text{ times}} &= \exp(\underbrace{a/b + \cdots + a/b}_{b \text{ times}}) \\ &= \exp(a) \\ &= e^a. \end{aligned}$$

This shows that $\exp(a/b) = e^{a/b}$: multiplying the number $\exp(a/b)$ by itself b times gives e^a.

Case 6. The final case is where x is an arbitrary real number. We have seen that $e^x = \exp(x)$ when x is an integer or a fraction. Is it true that $e^x = \exp(x)$ when x is an irrational number (those are the only real numbers we haven't considered). The problem here is: what is the definition of e^x when x is an irrational number? What does $e^{\sqrt{2}}$ or e^π mean? There is a good chance that you have not seen a definition of exponentiation by an irrational number. We make the following definition:

Definition. Let x be any real number, and let $e = \exp(1)$. Then we define e^x by

$$e^x := \exp(x).$$

The utility of this definition is that for cases 1–5, where we have a prior notion of the meaning of e^x, this definition agrees with the usual definition, but then it extends the meaning of exponentiation to the case of irrationals, as well.

To finish the story, we define exponentiation, in general:

Definition. Let a and x be real numbers with $a > 0$. Then

$$a^x := e^{x\ln(a)} = \exp(x\ln(a)).$$

Example. $2^\pi = e^{\pi\ln(2)} \approx 8.82$.

Some consequences of the definition:

- $a^{x+y} = a^x + a^y$.
- $(a^x)^y = a^{xy}$.
- $\dfrac{a^x}{a^y} = a^{x-y}$.
- $\ln(a^x) = x\ln(a)$.

Week 12, Monday: Differential equations.

Differential equations

Let y be a function of t. An equation in t and the derivates of y is called a *differential equation*. For instance, the following is a differential equation:

$$y'' - y = 0.$$

To make the dependence on t explicit, we might instead write

$$y''(t) - y(t) = 0.$$

Imagine that $y(t)$ gives the position of a particle on the real number line at time t. Then $y'(t)$ is the velocity of the particle—the rate of change of $y(t)$ over time, and $y''(t)$ is the acceleration of the particle—the rate of change of its velocity over time. The differential equation we are thinking about can be rewritten as

$$y''(t) = y(t).$$

So as the position of the particle gets larger, its acceleration increases.

To *solve* the differential equation, we need to find all functions $y(t)$ that satisfy the equation: what possible distance functions have this behavior. It turns out that the most general solution is

$$y = ae^t + be^{-t} \tag{32.1}$$

where a and b are any constants. To check this general y is a solution, we check that it satisfies the equation:

$$\begin{aligned}\left(ae^t + be^{-t}\right)'' - (ae^t + be^{-t}) &= \left(ae^t - be^{-t}\right)' - (ae^t + be^{-t}) \\ &= \left(ae^t + be^{-t}\right) - (ae^t + be^{-t}) \\ &= 0.\end{aligned}$$

We say there is a *two-parameter family of solutions*, the parameters being a and b. This turns out to be expected since the equation we are considering is a *second-order* differential equation, which means that the highest-order derivative in the equation is y'', a second derivative. To specify a particular solution, i.e., to determine the constants a and b, we need to set two initial conditions. For instance, suppose we want the solution with initial position $y(0) = 0$ and initial velocity $y'(0) = 1$. Plugging $y(0) = 0$ into equation (32.1), we get

$$0 = y(0) = ae^0 + be^{-0} = a + b,$$

Differentiating equation 32.1, we get

$$y'(t) = ae^t - be^{-t},$$

and hence, if $y'(0) = 1$, we need

$$1 = y'(0) = ae^0 - be^{-0} = a - b.$$

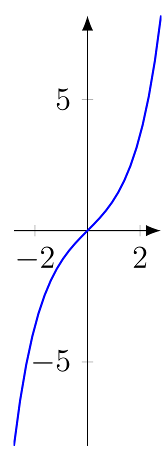
To summarize: the initial conditions force

$$\begin{aligned} a + b &= 0 \\ a - b &= 1. \end{aligned}$$

Adding the two equations gives $2a = 1$; so $a = 1/2$. Then since $a + b = 0$, we need $b = -1/2$. So the solution with initial position at the origin and initial velocity 1 is

$$y = \frac{1}{2}(e^t - e^{-t}).$$

The graph looks like this:



Graph of $y(t) = \frac{1}{2}(e^t - e^{-t})$.

Note that the differential equation and its initial conditions determine the movement of the particle, i.e., determine $y(t)$, for all time, not just stating from the initial time $t = 0$. The behavior is roughly like this: as t becomes large, the term e^{-t} becomes almost 0, so $y(t) \approx e^t/2$ for large t. This is what we might expect from the equation $y'' = y$. Once the position and velocity are positive, the acceleration is positive, which causes the speed to increase. The speed becomes even more positive, which causes the distance to increase, which causes the acceleration to increase, etc.

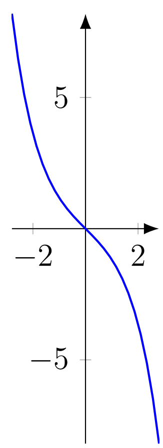
What if we had initial conditions $y(0) = 0$ and $y'(0) = -1$. This is just like before but now the initial velocity is to the left instead of to the right. Repeating the above reasoning, we find that we need to solve the system of equations

$$\begin{aligned} a + b &= 0 \\ a - b &= -1. \end{aligned}$$

Adding the equations gives $2a = -1$, this time; so $a = -1/2$. Then, since $a + b = 0$, we get $b = 1/2$. The solution is then

$$y = -\frac{1}{2}(e^t - e^{-t}).$$

The graph looks like this:



Graph of $y(t) = -\dfrac{1}{2}(e^t + e^{-t})$.

The particle speeds off in the negative direction.

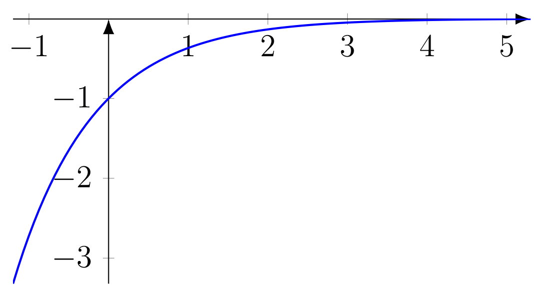
What would happen if the particle start at $y(0) = -1$, to the left of the origin, but with initial velocity $y'(0) = 1$, to the right. So the particle is moving to the origin, which according to $y'' = y$ should make the acceleration decrease. The set of equations for initial conditions $y(0) = -1$ and $y'(0) = 1$ is

$$\begin{aligned} -1 &= y(0) = ae^0 + b^{-0} = a + b \\ 1 &= y'(0) = ae^0 - b^{-0} = a - b \end{aligned}$$

or

$$\begin{aligned} a + b &= -1 \\ a - b &= 1. \end{aligned}$$

Adding the equations gives $2a = 0$, hence $a = 0$. Then $a + b = -1$ gives $b = -1$. So the solution in this case is $y(t) = -e^{-t}$.



Graph of $y(t) = -e^{-t}$.

The particle gets closer and closer to the origin, decelerating all the time and never quite getting there.

As a final example, imagine the initial condition $y(0) = y'(0) = 0$. So the particle is initially sitting at the origin with no velocity. We have $y''(t) = y$. So initially, we have $y''(0) = y(0) = 0$. So there is also no acceleration. The system of equations for the constants is

$$\begin{aligned} a + b &= 0 \\ a - b &= 0. \end{aligned}$$

Adding the equations gives $2a = 0$, hence, $a = 0$. Then the equation $a + b = 0$ implies $b = 0$, too. So the solution in this case is

$$y(t) = 0.$$

The particle just sits at the origin.

Next example. Now consider the first-order equation

$$ty'(t) = 3y(t).$$

Again, let's think of $y(t)$ of specifying the position of a particle on the real number line at time t. Suppose that $t \neq 0$ and solve for y':

$$y' = \frac{3y}{t}.$$

The t in the denominator is trying to decrease the velocity over time. On the other hand, the particle moving to the right would increase $y(t)$, which would increase the velocity. What is the resulting behavior?

This equation is known as a *separable* differential equation. That means that it is possible to get all of the ys on one side of the equation and the ts on the other:

$$\frac{y'}{y} = \frac{3}{t}.$$

Now try to integrate both sides with respect to t:

$$\int \frac{y'}{y}\, dt = \int \frac{3}{t}\, dt.$$

The right-hand side is easy:

$$\int \frac{1}{t}\, dt = 3\ln(t) + c$$

For the left-hand side, use the substitution $u = y$. Then $du = dy$, so we get

$$\int \frac{y'}{y}\, dt = \int \frac{du}{u}\, du = \ln(u) + \tilde{c} = \ln(y) + \tilde{c}.$$

Setting the two sides equal gives

$$\ln(y) = 3\ln(t) + k = \ln(t^3) + k$$

for some constant k. Exponentiate to get

$$e^{\ln(y)} = e^{\ln(t^3)+k} = e^k e^{\ln(t^3)},$$

and hence,

$$y = Kt^3.$$

The solution with initial condition $y(1) = 1$ is $y(t) = t^3$.

Week 12, Wednesday: Population models I.

Differential equations

As a warm-up, let's solve the differential equation

$$y' = \frac{3t}{y}.$$

This is a separable differential equation, meaning that we can get the ys on one side of the equality and the ts on the other:

$$yy' = 3t.$$

We can then solve by integrating both sides with respect to t:

$$\int y(t)y'(t)\,dt = \int 3t\,dt.$$

The right-hand side is

$$\int 3t\,dt = \frac{3}{2}t^2$$

(We will add a constant "$+c$" at the end of our calculations.) For the left-hand side, make the substitution $u = y(t)$. So $du = y'(t)\,dt$. Substituting gives:

$$\int y(t)y'(t)\,dt = \int u\,du = \frac{1}{2}u^2 = \frac{1}{2}y^2.$$

Setting the two sides equal and adding a constant gives the most general solution:

$$\frac{1}{2}y^2 = \frac{3}{2}t^2 + \tilde{c}$$

or, equivalently,

$$\boxed{y(t)^2 = 3t^2 + c}$$

for some constant c.

To find a particular solution, we can impose an initial condition. For instance, if $y(0) = 5$, then

$$25 = y(0)^2 = 3 \cdot 0^2 + c \quad \Rightarrow \quad c = 25,$$

and the solution is

$$y(t)^2 = 3t^2 + 25.$$

Exponential growth and decay model. Let $y(t)$ now denote the size of a population, varying over time. What happens if we assume that the rate of growth of the population is proportional to the size of the population? The rate of growth of the population is $y'(t)$ and the size of the population if $y(t)$. To say they are proportional is to say there is a constant k such that

$$y'(t) = ky(t).$$

This is a separable equation, which is easy to solve:

$$y'(t) = ky(t) \quad \Rightarrow \quad \frac{y'(t)}{y(t)} = k \quad \Rightarrow \quad \int \frac{y'(t)}{y(t)}\, dt = \int k\, dt.$$

The right-hand side is

$$\int k\, dt = kt.$$

The left-hand side can be solved with the u-substitution $u = y(t)$ and $du = y'(y)\, dt$:

$$\int \frac{y'(t)}{y(t)}\, dt = \int \frac{du}{u} = \ln(u) = \ln(y).$$

Setting these equal and adding a constant gives:

$$\ln(y) = kt + c.$$

Exponentiate both sides of this equation:

$$y = e^{\ln x} = e^{kt+c} = e^c e^{kt}.$$

Since e^c is just some constant, we will relabel it as a to get

$$\boxed{y(t) = ae^{kt}.}$$

Setting $t = 0$, we see

$$y(0) = ae^0 = a.$$

Hence, a is the initial population.

Example. If $y(t) = ae^{kt}$, at what time t has the population doubled?

SOLUTION: The initial population size is a. So we are trying to find the time t when $y(t) = 2a$, so we need to solve

$$ae^{kt} = 2a.$$

Supposing that $a > 0$, we need to solve

$$y(t) = e^{kt} = 2$$

for t. Take logs:

$$\ln(2) = \ln(e^{kt}) = kt.$$

Hence, assuming $k \neq 0$,

$$t = \frac{\ln(2)}{k}.$$

Population model based on Newton's law of cooling. Suppose now that the rate of change of the population is governed by the differential equation

$$y'(t) = r(S - y(t))$$

where k and S are positive constants.

Questions:

1. When is the population increasing? Decreasing?

   ANSWER: We have

   $$y'(t) = r(S - y(t)) > 0 \quad \Leftrightarrow \quad S - y(t) > 0 \quad \Leftrightarrow \quad S > y(t).$$

   So the population is increasing whenever it's less thatn S and decreasing whenever it's larger than S.

2. What is the long-term behavior of the population?

   ANSWER: Given the answer to the previous problem it seems like the population should stabilize at S

3. Solve the equation.

SOLUTION: The equation is separable:

$$y'(t) = r(S - y(t)) \quad \Rightarrow \quad \frac{y'(t)}{S - y(t)} = r$$

$$\Rightarrow \quad \int \frac{y'(t)}{S - y(t)}\, dt = \int r\, dt$$

$$\Rightarrow \quad \int \frac{y'(t)}{S - y(t)}\, dt = rt + c$$

Substitute $u = S - y(t)$. Then $du = -y'(t)\, dt$. So

$$\int \frac{y'(t)}{S - y(t)}\, dt = -\int \frac{du}{u} = -\ln(u) = -\ln(S - y(t)) = \ln\left((S - y(t))^{-1}\right).$$

Therefore,

$$\ln\left(\frac{1}{S - y(t)}\right) = rt + c.$$

Exponentiate:

$$\frac{1}{S - y(t)} = e^{rt+c} = e^c e^{rt} = ae^{rt} \qquad (a = e^c),$$

and solve for $y(t)$:

$$\frac{1}{S - y(t)} = ae^{rt} \quad \Rightarrow \quad S - y(t) = \frac{1}{ae^{rt}}$$

$$\Rightarrow \quad y(t) = S - \frac{1}{ae^{rt}}.$$

Therefore, the solution is

$$y(t) = S - \frac{1}{a}e^{-rt}.$$

Note that $y(t) \to S$ as $t \to \infty$.

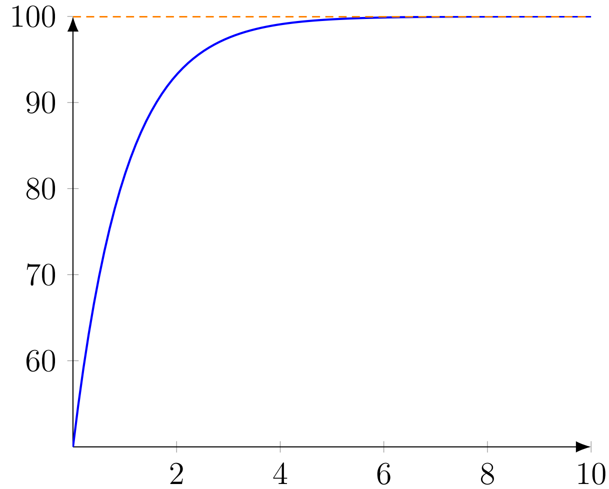
Let's now make the initial population explicit in the solution. Say I is the initial population. Then

$$I = y(0) = S - \frac{1}{a}e^0 = S - \frac{1}{a}.$$

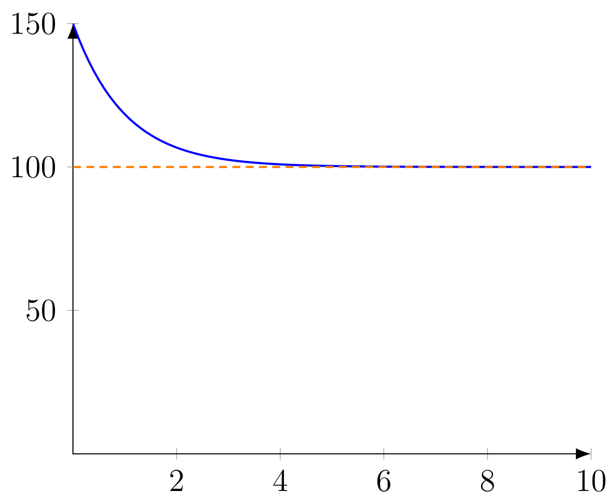
Therefore, $a = (S - I)^{-1}$. Our final form for the equation is

$$\boxed{y(t) = S - (S - I)e^{-rt},}$$

where $I = y(0)$ is the initial population.



Graph of $y(t) = S - (S - I)e^{-rt}$ with $S = 100$, $I = 50$, and $r = 1$.



Graph of $y(t) = S - (S - I)e^{-rt}$ with $S = 100$, $I = 150$, and $r = 1$.

Week 13, Monday: Population models II.

Logistic growth model. Let $P(t)$ be the size of a population at time t. The logistic growth model is the differential equation

$$P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right).$$

It says the growth in population is proportional to the size of the existing population with an extra factor to account for limited resources. When the population is small (when P is much smaller then K), we see $P' \approx rP$, which we've already seen leads to exponential growth. However, as P gets close to K over time, the factor $1 - P/K$ slows the growth.

Solution. The equation is separable and can be solved using integration using the technique of partial fractions.

$$P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right) \quad \Rightarrow \quad \frac{P'(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)} = r.$$

The technique of partial fractions requires us to find constants A and B such that

$$\frac{1}{P(t)\left(1 - \frac{P(t)}{K}\right)} = \frac{A}{P(t)} + \frac{B}{1 - \frac{P(t)}{K}}. \tag{34.1}$$

We have

$$\frac{A}{P(t)} + \frac{B(t)}{1 - \frac{P(t)}{K}} = \frac{A\left(1 - \frac{P(t)}{K}\right) + BP(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)}. \tag{34.2}$$

Comparing numerators in equations (34.1) and (34.2), we need to adjust A and B so that

$$1 = A\left(1 - \frac{P(t)}{K}\right) + BP(t).$$

Or, rearranging:

$$1 = A + \left(-\frac{A}{K} + B\right) P(t).$$

We get an equality if

$$A = 1 \quad \text{and} \quad -\frac{A}{K} + B = 0.$$

So $A = 1$ and $B = 1/K$. Therefore we can write (double-check!):

$$\frac{1}{P(t)\left(1 - \frac{P(t)}{K}\right)} = \frac{1}{P(t)} + \frac{1/K}{1 - \frac{P(t)}{K}}. \tag{34.3}$$

Back to solving the differential equation:

$$\frac{P'(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)} = r \quad \Rightarrow \quad \int \frac{P'(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)}\, dt = \int r\, dt$$

$$\Rightarrow \quad \int \frac{P'(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)}\, dt = rt + \text{constant}.$$

For the left-hand side, use equation (34.3):

$$\begin{aligned}
\int \frac{P'(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)}\, dt &= \int \frac{P'(t)}{P(t)} + \frac{P'(t)/K}{1 - \frac{P(t)}{K}}\, dt \\
&= \int \frac{P'(t)}{P(t)}\, dt + \frac{1}{K} \int \frac{P'(t)}{1 - \frac{P(t)}{K}}\, dt \\
&= \ln P(t) + \frac{1}{K} \int \frac{P'(t)}{1 - \frac{P(t)}{K}}\, dt
\end{aligned}$$

For the remaining integral, let $u = 1 - P(t)/K$. Then $du = -\frac{1}{K} P'(t)\, dt$, and $-K\, du = P'(t)\, dt$. Therefore,

$$\begin{aligned}
\frac{1}{K} \int \frac{P'(t)}{1 - \frac{P(t)}{K}}\, dt &= \frac{1}{K} \int \frac{-K}{u}\, du \\
&= -\int \frac{du}{u}
\end{aligned}$$

$$= -\ln(u) + \text{constant}$$

$$= -\ln\left(1 - \frac{P(t)}{K}\right) + \text{constant}.$$

Putting this all together:

$$\ln P(t) - \ln\left(1 - \frac{P(t)}{K}\right) = \ln P(t) + \ln\left(\left(1 - \frac{P(t)}{K}\right)^{-1}\right)$$

$$= \ln\left(P(t)\left(1 - \frac{P(t)}{K}\right)^{-1}\right)$$

$$= rt + \text{constant}.$$

Exponentiate both sides to get

$$P(t)\left(1 - \frac{P(t)}{K}\right)^{-1} = e^{rt}e^{\text{constant}} = ae^{rt}$$

for some positive constant a. We now need to solve this equation for $P(t)$:

$$ae^{rt} = P(t)\left(1 - \frac{P(t)}{K}\right)^{-1} = \frac{KP(t)}{K - P(t)}$$

$$\Rightarrow ae^{rt}(K - P(t)) = KP(t)$$

$$\Rightarrow aKe^{rt} = ae^{rt}P(t) + KP(t) = (ae^{rt} + K)P(t)$$

$$\Rightarrow P(t) = \frac{aKe^{rt}}{ae^{rt} + K}$$

$$\Rightarrow P(t) = \frac{aK}{a + Ke^{-rt}}.$$

We would like to express the arbitrary constant a in terms of the initial population:

$$P(0) = \frac{aKe^0}{ae^0 + K} = \frac{aK}{a + K}$$

$$\Rightarrow P(0)(a + K) = aK$$

$$\Rightarrow P(0)K = aK - P(0)a = a(K - P(0))$$

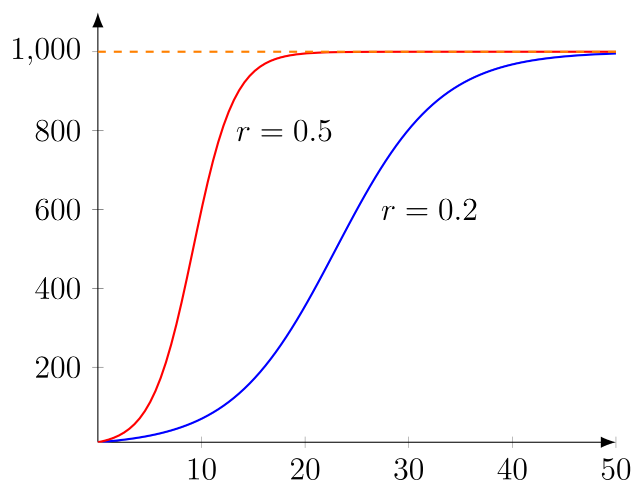
$$a = \frac{P(0)K}{K - P(0)}.$$

Substituting this expression for a an simplfying gives the final form for the solution

$$\boxed{P(t) = \frac{P(0)K}{P(0) + (K - P(0))e^{-rt}}.}$$

It's easy to see from this equation that the limiting population is

$$\lim_{t\to\infty} P(t) = K.$$



Graph of $P(t)$ with $K = 1000$ and $P(0) = 10$ and two different growth rates: $r = 0.5$ in red and $r = 0.2$ in blue.

Exercise (adapted from our text). A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the refuge can support no more than 4000 elk. Use a logistic model to predict the elk population in 15 years.

SOLUTION: The carrying capacity is $K = 4000$, so the logistic model in this situation is

$$P'(t) = rP(t)\left(1 - \frac{P(t)}{4000}\right)$$

where we can determine r from the additional information we're given. The initial population size is $P(0) = 40$. From the solution to the logistic equation we derived above, we have

$$\begin{aligned} P(t) &= \frac{4000P(0)}{P(0) + (4000 - P(0))e^{-rt}} \\ &= \frac{160000}{40 + 3960e^{-rt}} \\ &= \frac{4000}{1 + 99e^{-rt}} \end{aligned}$$

We are given that $P(5) = 104$. Therefore,

$$104 = P(5) = \frac{4000}{1 + 99e^{-5r}}.$$

Solve for r:

$$\begin{aligned} 104 = \frac{4000}{1 + 99e^{-5r}} \quad &\Rightarrow \quad 104(1 + 99e^{-5r}) = 4000 \\ &\Rightarrow \quad e^{-5r} = \frac{1}{99}\left(\frac{4000}{104} - 1\right) = \frac{487}{1287} \\ &\Rightarrow \quad -5r = \ln\left(\frac{487}{1287}\right) \\ &\Rightarrow \quad r \approx 0.194. \end{aligned}$$

So our model for this population is

$$P(t) = \frac{4000}{1 + 99e^{-0.194\,t}}$$

So we would predict the population after 15 years to be

$$P(15) = \frac{4000}{1 + 99e^{-0.194 \cdot 15}} \approx 626.$$

Week 13, Wednesday: Lokta-Volterra model.

Lotka-Volterra predator-prey model

The Lotka-Volterra model is a model of population growth for competing species. It was first proposed by Lotka in 1920 and used in his book on biomathematics. For Volterra's involvement, here is a quote from Wikipedia:

> The same set of equations were published in 1926 by Vito Volterra, a mathematician and physicist who had become interested in mathematical biology. Volterra's enquiry was inspired through his interactions with the marine biologist Umberto D'Ancona who was courting his daughter at the time and later was to become his son-in-law. D'Ancona studied the fish catches in the Adriatic Sea and had noticed that the percentage of predatory fish caught had increased during the years of World War I (1914–18). This puzzled him as the fishing effort had been very much reduced during the war years. Volterra developed his model independently from Lotka and used it to explain d'Ancona's observation.

Here is a sample of Anacona's data: The table shows percentages of predators found in a certain fish catch:

year	1914	1915	1916	1917	1918	1919	1920	1921	1922	1923
percentage	12	21	22	21	36	27	16	16	15	11

What might account for these data?

$$x(t) = \text{ prey fish population}$$
$$y(t) = \text{ predator fish population}$$

The Lotka-Volterra equations are

$$x'(t) = x(t)(a - by(t))$$

$$y'(t) = y(t)(-c + dx(t)).$$

where a, b, c, d are positive constants. We also suppose that $x(t)$ and $y(t)$ are always positive.

Exercises.

1. What type of growth for $x'(t)$ is predicted if $y(t)$ is very small? What type of growth for $y'(t)$ is predicted if $x(t)$ is very small? How do the parameters a and c influence that growth?

2. Consider the system of equations when $a = 2$, $b = 1$, $c = 0.25$, and $d = 1$:

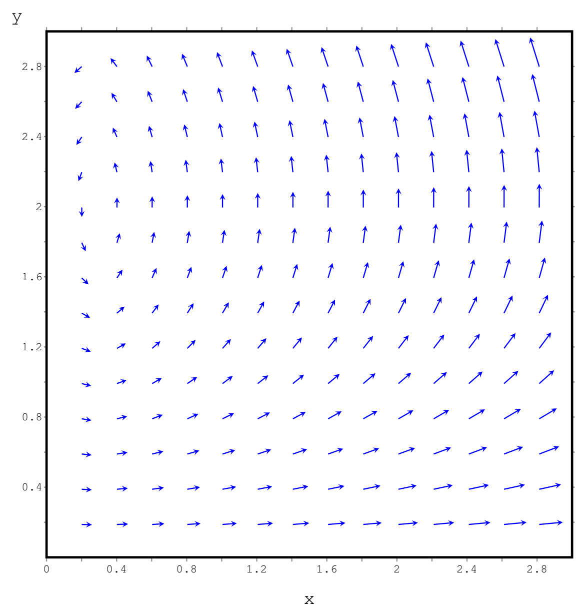
$$x'(t) = x(t)(2 - y(t))$$
$$y'(t) = y(t)(-0.25 + x(t)).$$

   Under what condition is $x'(t) = 0$? Under what condition is $y'(t) = 0$. How about bot $x'(t) = 0$ and $y'(t) = 0$? What does this mean in terms of the two populations?

3. Answer the previous question of the general system (with arbitrary positive constants a, b, c, d).

Picture the Lotka-Volterra system. Here is a way of picturing the system:

$$x'(t) = x(t)(2 - y(t))$$
$$y'(t) = y(t)(-0.25 + x(t)).$$



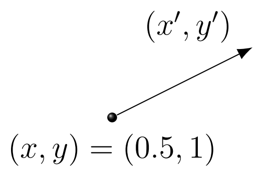
In the diagram, the horizontal axis is the prey population, x, and the vertical axis is the predator population, y. Each point (x, y) in the plane represents a potential state of the two-species system. For instance, the point $(0.5, 1)$ represents the state in which the prey population is 0.5 and the predator population is 1. (To make this more realistic, we could think of the units as being hundreds of fish.) What do the arrows mean? At each point (x, y), we have attached the vector

$$(x', y') = (x(t)(2 - y(t)),\ y(t)(-0.25 + x(t))).$$

For example, if $x = 0.5$ and $y = 1$, then

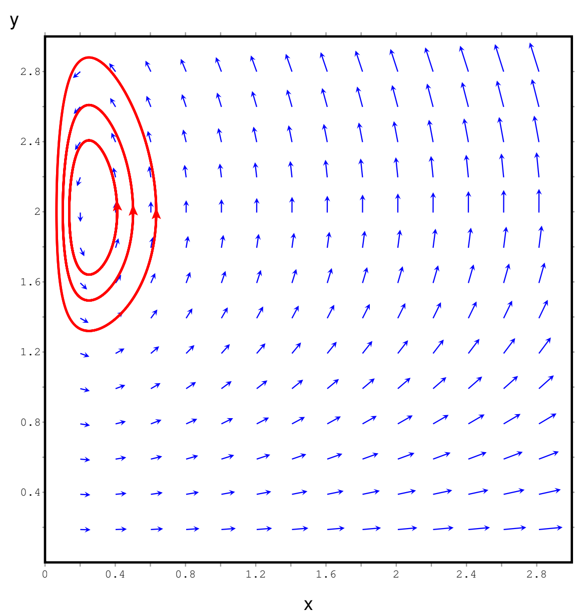
$$\begin{aligned} x' &= x(2 - y) = 0.5(2 - 1) = 0.5 \\ y' &= y(-0.25 + x) = 1(-0.25 + 0.5) = 0.25. \end{aligned}$$

So the rate of change of the prey population is 0.5 and the rate of change of the predatory population is 0.25. We picture this in the diagram as follows:



Exercise. Find the point $(0.5, 1)$ in the diagram and check that the arrow attached to that point matches the picture given above.

The below illustrates three possible solutions to the system of differential equations. The idea is to think of the system of arrows as a flowing liquid. Drop a tiny boat in the fluid and see where it flows over time. The three solutions correspond to three different initial conditions, i.e., three spots where we have dropped a tiny boat.



If you follow just the x coordinate, you will see how the prey population changes over time. If you follow the y coordinate, you will see how the pedator population changes over time.

Exercises.

1. How do the populations change over time for the three different solutions?
2. Where is the point where $x' = y' = 0$ in the picture?

General question. What are some of the unrealistic assumptions behind the model?

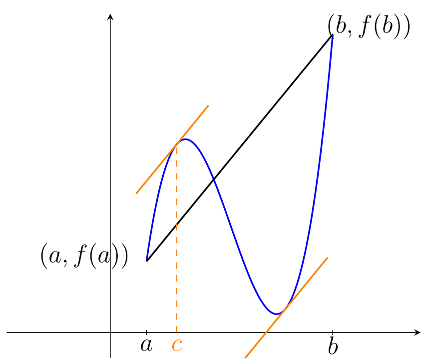
Week 13, Friday: Mean Value Theorem (MVT).

Mean Value Theorem

Suppose you drive through two toll booths that are 80 miles apart. Your time is recorded at each booth, and it is determined that it took you 1 hour to travel that distance. Why would it be reasonable for you to be issued a ticket? The mathematical basis for your answer is the following theorem:

Mean Value Theorem (MVT). Let f be a continuous function on $[a, b]$ and differentiable on (a, b). Then there exists a number c with $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Mean value theorem (with two possible choices for c).

Proof. Math 112. □

If we think of f as a distance function, the MVT says that at some point between a and b, the instantaneous speed is equal to the average speed over the time interval $[a, b]$.

The MVT is a *local-to-global* tool: the instantaneous speed only depends on the behavior of f near c, while the average speed depends on the net behavior of f over the entire interval. In more detail, recall the definition of the derivative:

$$f'(c) = \lim_{x \to \infty} \frac{f(c+h) - f(c)}{h}.$$

If you think of the definition of the limit, it's clear that $f'(c)$ only depends on the behavior of f in a tiny interval around c.

Fundamental applications of the MVT. Consider the following statement:

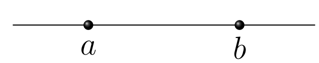
> If $f'(0) = 0$, then f is a constant function.

How would you go about proving this from the definition of the derivative? Note that it relates local behavior to global behavior: speed 0 to being constant everywhere, which suggest the MVT might be involved. Another similar statement: if $f'(x) > 0$ for all x in an interval, then f is increasing on that interval. We will prove these now as corollaries of the MVT.

Corollary. Let f be a differentiable function on an open interval I. Then:

1. If $f'(x) = 0$ for all x in I, then f is constant on I.
2. If $f'(x) > 0$ for all x in I, then f is increasing on I.
3. If $f'(x) < 0$ for all x in I, then f is decreasing on I.

Proof. Let a and b be in the interval I, and suppose $a < b$:

$I =$ 

By the MVT, there exists c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{36.1}$$

1. If $f' = 0$ at all points in I. Then, in particular, $f'(c) = 0$. Therefore, equation (36.1) becomes
$$0 = \frac{f(b) - f(a)}{b - a},$$
which implies $f(a) = f(b)$. Since a, b are arbitrary points in I, it follows that f is constant on I.

2. If $f' > 0$ at all points in I, then equation (36.1) becomes

$$0 < f'(c) = \frac{f(b) - f(a)}{b - a},$$

   which implies $f(a) < f(b)$. Since a, b are arbitrary points in I, it follows that f is increasing on I.

3. If $f' < 0$ at all points in I, a similar argument shows that f is decreasing on I. □

Addendum. That finishes the main point of today's lecture. With the extra time we have, let's take a look at a technical point having to do with u-substitutions.

Example. Use a u-substitution to compute the following integral:

$$\int_0^1 x^2(x^3 + 1)^3\, dx.$$

Solution. Let $u = x^3 + 1$. Then $du = 3x^2\, dx$. Therefore,

$$\int x^2(x^3 + 1)^3\, dx = \int \frac{1}{3}u^3\, du = \int \frac{1}{12}u^4 + c = \frac{1}{12}(x^3 + 1)^4 + c.$$

It follows that

$$\begin{aligned}\int_0^1 x^2(x^3 + 1)^3\, dx &= \left(\frac{1}{12}(x^3 + 1)^4\right)\Bigg|_{x=0}^{1} \\ &= \frac{1}{12}((1^3 + 1)^4 - (0^3 + 1)^4) \\ &= \frac{1}{12}(2^4 - 1^4) \\ &= \frac{15}{12} = \frac{5}{4}.\end{aligned}$$

An alternative is to note that when $x = 0$, we have $u = 1$ and when $x = 1$, we have $u = 2$. Therefore,

$$\int_0^1 x^2(x^3 + 1)^3\, dx = \int_1^2 \frac{1}{3}u^3\, du = \frac{1}{12}u^4\Big|_{u=1}^{2} = \frac{1}{12}(2^4 - 1^4) = \frac{5}{4}.$$

The point: When evaluating a definite integral using u-substitutions, be careful with the limits of integration.

Week 14, Monday: Proof of the fundamental theorem of calculus.

The Fundamental Theorem of Calculus

The fundamental theorem of calculus is what allows us to compute integrals (areas under graphs) using antiderivatives. Some examples:

1. $\displaystyle\int_0^1 2x\,dx = \int_0^1 (x^2)'\,dx = x^2\,\Big|_0^1 = 1^2 - 0^2 = 1.$

2. $\displaystyle\int_0^2 x^2\,dx = \int_0^2 \left(\frac{1}{3}x^3\right)'\,dx = \frac{1}{3}x^3\,\Big|_0^2 = \frac{1}{3}2^3 - \frac{1}{3}0^3 = \frac{8}{3}.$

3. $\displaystyle\int_0^{\pi/2} \cos(x)\,dx = \int_0^{\pi/2} (\sin(x))'\,dx = \sin(x)\,\Big|_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1.$

Our goal today is to prove the fundamental theorem of calculus. Before attempting to understand the proof, the reader should review the definition of the integral and the mean value theorem, included in the addendum.

Theorem. Suppose f is integrable on $[a,b]$ and there exists g such that $f' = g$. Then

$$\int_a^b f = g(b) - g(a), \quad \text{i.e.,} \quad \int_a^b g' = g(b) - g(a).$$

Proof. Let $P = \{t_0, \ldots, t_n\}$ be any partition of $[a,b]$. Apply the mean value theorem to g on each subinterval of P. For each $i = 1, \ldots, n$, we get $c_i \in [t_{i-1}, t_i]$ such that

$$g'(c_i) = \frac{g(t_i) - g(t_{i-1})}{t_i - t_{i-1}}.$$

Since $g'(c_i) = f(c_i)$, we can substitute and rearrange to get

$$f(c_i)(t_i - t_{i-1}) = g(t_i) - g(t_{i-1}), \tag{37.1}$$

for each i.

Let $M_i = \text{lub}\, f([t_{i-1}, t_i])$ and $m_i = \text{glb}(f([t_{i-1}, t_i])$, as usual. It follows that

$$m_i \le f(c_i) \le M_i \quad \Rightarrow \quad m_i(t_i - t_{i-1}) \le f(c_i)(t_i - t_{i-1}) \le M_i(t_i - t_{i-1}),$$

for each i. Summing over i,

$$L(f, P) \le \sum_{i=1}^{n} f(c_i)(t_i - t_{i-1}) \le U(f, P).$$

Recall that $\sum_{i=1}^{n} f(c_i)(t_i - t_{i-1})$ means

$$f(c_1)(t_1 - t_0) + f(c_2)(t_2 - t_1) + + \cdots + f(c_n)(t_n - t_{n-1}).$$

Using equation (37.1),

$$L(f, P) \le \sum_{i=1}^{n} (g(t_i) - g(t_{i-1})) \le U(f, P).$$

But $\sum_{i=1}^{n}(g(t_i) - g(t_{i-1}))$ is a *telescoping sum* adding up to $g(b) - g(a)$:

$$\begin{aligned}\sum_{i=1}^{n}(g(t_i) - g(t_{i-1})) &= (g(t_1) - g(t_0)) + (g(t_2) - g(t_1)) + (g(t_3) - g(t_2)) + \\ &\qquad \cdots + (g(t_{n-1}) - g(t_{n-2})) + (g(t_n) - g(t_{n-1})) \\ &= g(t_n) - g(t_0) \\ &= g(b) - g(a).\end{aligned}$$

Thus,

$$L(f, P) \le g(b) - g(a) \le U(f, P)$$

for each partition P. It follows that

$$L(f) = \text{lub}_P\{L(f, P)\} \le g(b) - g(a) \le \text{glb}_P\{U(f, P)\} = U(f).$$

(The P in the subscript means we are considering the set of lower or upper sums for all possible partitions.) However, since f is integrable,

$$L(f) = U(f) = \int_a^b f.$$

Therefore, $\int_a^b f = g(b) - g(a)$. □

Addendum

Definition of the integral. Let f be a bounded function on a closed interval $[a,b]$. (The word "bounded" here means that there is some constant B such that $-B \le f(x) \le B$ for all $x \in [a,b]$.) A *partition* of $[a,b]$ is a set $P = \{t_0, t_1, t_2, \dots, t_n\}$ with each $t_i \in [a,b]$ and with $t_0 = a$ and $t_n = b$. By convention, we always take $t_0 < t_1 < \cdots < t_n$. In that case the interval $[t_{i-1}, t_i]$ is called the *i-th subinterval of P*. The values the function f takes on the i-th interval is denoted $f([t_{i-1}, t_i])$:

$$f([t_{i-1}, t_i]) = \{f(x) : t_{i-1} \le x \le t_i\}.$$

This set is called the *image of* $[t_{i-1}, t_i]$ under f. For each $i = 1, \dots, n$ define

$$m_i = \operatorname{glb} f([t_{i-1}, t_i]),$$
$$M_i = \operatorname{lub} f([t_{i-1}, t_i]),$$

then define the *lower and upper sums for f with respect to P*:

$$L(f,P) = \sum_{i=1}^{n} m_i (t_i - t_{i-1}),$$
$$U(f,P) = \sum_{i=1}^{n} M_i (t_i - t_{i-1}).$$

The *upper and lower integrals* for f are:

$$L\int_a^b f = \operatorname{lub}\{L(f,P) : P \text{ a partition of } [a,b]\}$$
$$U\int_a^b f = \operatorname{glb}\{U(f,P) : P \text{ a partition of } [a,b]\}.$$

The function f is *integrable* if

$$L\int_a^b f = U\int_a^b f,$$

and in that case, the common value is the *integral of f on* $[a,b]$, denoted $\int_a^b f$ or $\int_a^b f(x)\,dx$. □

Definition of a Riemann sum. Pick real numbers c_i such that $t_{i-1} \le c_i \le t_i$ for $i = 1, \dots, n$. The quantity

$$R = \sum_{i=1}^{n} f(c_i)(t_i - t_{i-1})$$

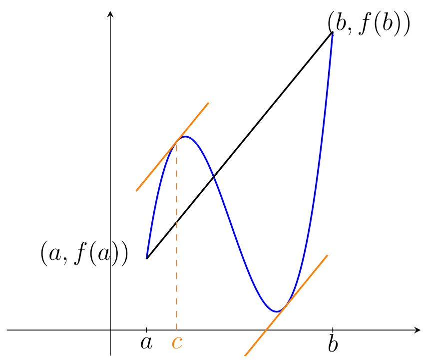
is called a *Riemann sum* for f relative to the partition P. Note that

$$L(f,P) \le R \le U(f,P)$$

for all choices of the c_i. Thus, a Riemann sum estimates the integral of f, provided the integral exists.

The Mean value theorem (MVT). Let f be a continuous function on $[a,b]$ and differentiable on (a,b). Then there exists a number c with $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Mean value theorem (with two possible choices for c).

Homework assignments

Week 2, Friday

1. Exercises from Section 1.2: 8, 10, 12, 14. In exercise 8, you'll need a calculator of some sort. Remember that exercise 8 also asks you to use your calculations to guess the limit.

2. Give an ε-δ proof that $\lim_{x\to 3} 6x - 4 = 14$. Try to emulate the similar proofs given in class as closely as possible. Make sure that you proof consists of complete sentences!

3. Let $f(x) = x^3$. Give an ε-δ proof that $\lim_{x\to 0} f(x) = 0$. Do not use the limit theorems given in class. (Hint: start by letting $\varepsilon > 0$. You now need to find $\delta > 0$ such that $0 < |x - 0| < \delta$ implies $|f(x) - 0| < \varepsilon$. In other words, you want $0 < |x| < \delta$ to imply $|x^3| < \varepsilon$. Which δ works? Once you've figured that out, try to emulate the write-up for the proof you gave in previous problem as closely as possible.)

Week 3, Tuesday

1. Give an ε-δ proof that $\lim_{x\to 0}\sqrt{x} = 0$. (Since the square root function only makes sense for nonnegative numbers, there is an unwritten assumption that when you impose the condition $0 < |x - 0| < \delta$, you are only considering positive x. Thus, the condition is equivalent to $0 < x < \delta$, without absolute value signs.) In your proof, please try to follow the examples in class as closely as possible, always using complete sentences.

2. Section 1.3: 8, 16, 30, 38(abcd). (For this problem, you don't need to explain your reasoning.)

3. Use our limit theorem to find
$$\lim_{x\to 2} \frac{x^2+4}{x^2-3x}.$$
For full credit, mimic the proof given in Example 2 of the lecture for Friday of week 2, justifying each step by referring to parts 1–3 of the limit theorem or to "prop", meaning a proposition in that lecture.

Week 3, Friday

1. Compute the following limits using the cancellation trick, the rationalization trick, and the limit theorem. Show all of the steps of your calculation. (**Note:** Be careful not to drop $\lim_{x\to c}$ across equal signs until the last step.)

   (a) $\displaystyle\lim_{x\to 1}\frac{x^3-x}{x-1}$.

   (b) $\displaystyle\lim_{x\to 0}\frac{x}{x^2-x}$.

   (c) $\displaystyle\lim_{x\to 2}\frac{2x^2-x-3}{x+1}$.

   (d) $\displaystyle\lim_{x\to 2}\frac{x^3-8}{x-2}$.

   (e) $\displaystyle\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$.

   (f) $\displaystyle\lim_{x\to 16}\frac{4-\sqrt{x}}{x-16}$.

2. Suppose that $\lim_{x\to 0} f(x) = 1$. Use the definition of the limit with $\varepsilon = 1$ to show that there must be some open interval about 0 such that $f(x) > 0$ for every x in that interval, except possibly at $x = 0$.

Week 4, Tuesday

1. **Graph** the following functions and **compute** the limits (see last Friday's lecture notes and section 1.4 of our text):

   (a) $\lim_{x\to 1+} f(x)$ where $f(x) = \begin{cases} x & \text{if } x \leq 1 \\ 1-x & \text{if } x > 1. \end{cases}$

   (b) $\displaystyle\lim_{x\to 3^-} \frac{1}{3-x}$.

2. Define

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 3x-2 & \text{if } x > 1. \end{cases}$$

   (a) Graph g.

   (b) Evaluate $\lim_{x\to 1} g(x)$ by computing both $\lim_{x\to 1^+} g(x)$ and $\lim_{x\to 1^-} g(x)$. (Recall that a limit exists at some point $x = c$ if and only if both the right- and left-hand limits exist at c and are equal.)

   (c) Why isn't g continuous at 1?

   (d) What part of the definition of the limit allows $\lim_{x\to 1} g(x)$ to exist even when g is not continuous at 1?

3. Evaluate the derivative of $f(x) = 3 - x^2$ at $x = 2$ by evaluating the limit

$$\lim_{h\to 0} \frac{f(2+h) - f(2)}{h}.$$

   (Show your work, as usual.)

Week 4, Friday

1. Let $f(t) = 3t^2 - t + 2$, and think of f as describing the motion of a particle along the y=axis.

   (a) What is the average speed of the particle between the times $t = 2$ and $t = 4$?

   (b) Use the definition of the derivative to compute $f'(2)$. (Show your work in computing the required limit. Be careful not to drop the limit sign across equalities until the last step in your computation.)

   (c) Find the equation for the tangent line for f at $t = 2$.

2. Suppose f is a differentiable function whose derivative has the following properties:

   - $f'(x) > 0$ for $x < 1$,
   - $f'(x) = 0$ for $x = 1$,
   - $f'(x) > 0$ for $1 < x < 2$,
   - $f'(x) = 0$ for $x = 2$,
   - $f'(x) > 0$ for $x > 2$.

   Sketch a possible graph of f.

3. The derivative of $\cos(x)$ is $\sin(x)$, i.e., $\cos'(x) = -\sin(x)$.

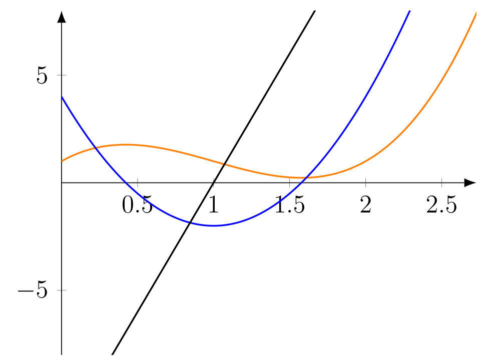
   (a) Use the product rule to compute the derivative of $x^3 \cos(x)$.

   (b) Use the quotient rule to compute the derivative of $\dfrac{x^5}{\cos(x)}$.

Week 5, Tuesday

1. Use the definition of the derivative to compute the derivative of $f(x) = 1/x^2$.

2. The figure below contains the graphs of three functions: $f(x)$, its derivative $f'(x)$, and the derivative of $f'(x)$, denoted $f''(x)$. Which is which?



Graph of $f(x)$, $f'(x)$, and $f''(x)$.

3. Let $f(x) = x^2 - 2x + 3$.

   (a) Compute $f'(x)$ using our derivative theorem in several steps as in the examples we did in class illustrating the use of the derivative theorem, using just the facts that the derivative of a constant is 0 and $(x)' = 1$.

   (b) (i) For which values of x is $f'(x) < 0$? What does that mean about the graph of f? (ii) For which value of x is $f'(x) = 0$? What does that mean about the graph of f? (iii) For which values of x is $f'(x) > 0$? What does that mean about the graph of (x)?

   (c) Use the information about the graph you've just gathered from the derivative to plot $f(x)$. Label the points where the graph hits the y-axis and where the derivative is 0 with their coordinates.

4. Compute the derivatives of the following polynomials. You do not need to show your work—use this exercise as an excuse to learn how to take the derivative of a polynomial in your head.

   (a) $4x^3 - 2x^2 + 5x + 6$.

   (b) $x^5 + 3x^4 + x^2 + 3x + 1$.

   (c) $x^{16} + 4x^4 + 12$.

5. Here are some derivatives of common functions:

$$(\ln(x))' = \frac{1}{x}, \quad (e^x)' = e^x, \quad (\sin(x))' = \cos(x), \quad (\cos(x))' = -\sin(x).$$

Use these facts and our derivative theorem (i.e., the sum, product, and quotient rules) to compute the derivatives of the following functions. Show your work.

(a) $x \ln(x)$.

(b) $\cos(x) + x^4 \sin(x)$.

(c) $e^x \sin(x) + x^2$.

Week 5, Friday

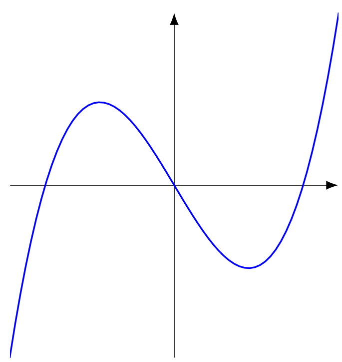
1. Use linearity, the product rule, the quotient rule, and the chain rule to evaluate the following derivatives. See the essential derivatives handout and the summary on page 136 of our text for help. Show your work.

(a) $x^2 \sin(x)$ (b) $\frac{x+2}{x-5}$ (c) $\sin(x)\cos(x)$

(d) $\sqrt{2x^3 + \ln x}$ (e) $\sec(x^2 + 5)$ (f) e^{5x^2+2x+1}

(g) $\cos^2(x) + \sin^2(x)$ (h) $e^{\tan x}$ (i) $\ln(\sin(x^5))$.

2. The graph of a function f is show below. Draw this graph and overlay it with the graph of f' so that we can see the relation between the two.



Graph of a function.

3. Compute the equation of the tangent line to $f(x) = \sin(x)$ at the point $x = 3\pi/4$. Draw the graph of $f(x)$ and this tangent line. Show your work.

Week 6, Tuesday

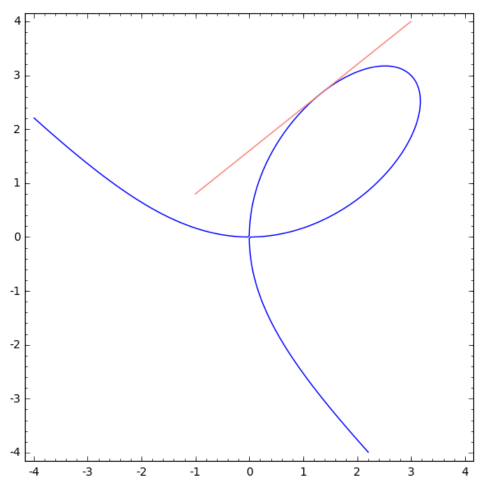
1. Compute the derivatives (again using the essential derivatives handout, if necessary).

   (a) $(6x+5)^7(x+4)^5$ (b) $\sqrt{x}\sin(x)$ (c) $\cos(x^4+3x+4)$.

2. Consider the curve in the plane defined as the set of points satisfying

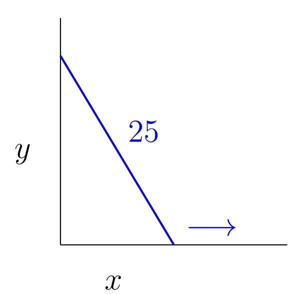
$$x^3 + y^3 - 6xy = 0.$$

   The curve is pictured below:



   The point $(4/3, 8/3)$ sits on the curve (i.e., $x = 4/3$ and $y = 8/3$ satisfies the equation). Find the equation of the tangent line at $(4/3, 8/3)$.

3. A 25-foot long blue ladder is leaning against the wall of a building, as pictured below. The base of the ladder is pulled away from the wall at a rate of 2 feet per second.



(a) How fast is the top of the ladder moving down the wall when its base is 7 feet from the wall?

(b) Let A be the area of the triangle formed by the walls and the ladder. Find the rate at which A is changing when the base of the ladder is 7 feet from the wall.

Week 6, Friday

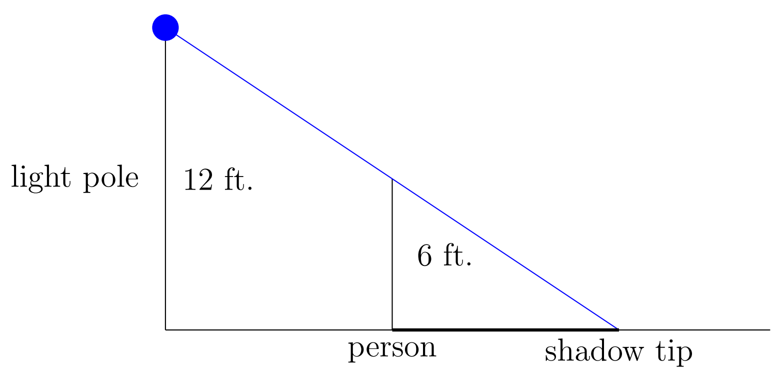
1. A person who is 6 feet tall walks at a rate of 4 feet per second away from a light that is 12 feet above the ground as shown below.

   (a) At what rate is the tip of the person's shadow moving when the person is k feet from the light?

   (b) As time goes on, is that rate constant? increasing? decreasing?

   (c) At what rate is the length of the shadow increasing when the person is k feet from the light?

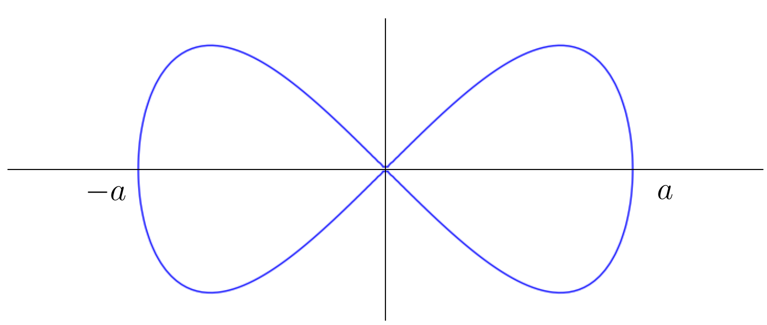
   (d) As time goes on, is that rate constant? increasing? decreasing?



2. Consider the figure eight curve shown on the next page whose points (x, y) satisfy

$$x^4 = a^2(x^2 - y^2)$$

for some $a > 0$:



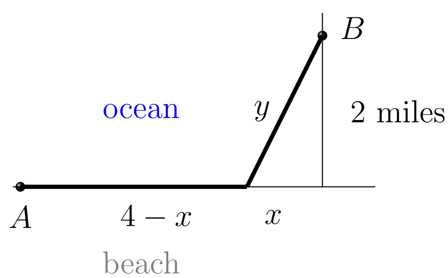
Find the coordinates (in terms of a) of the four points on the curve where the tangent line is horizontal. (You can use the fact that the slope is not horizontal at the origin.)

Week 7, Friday

1. After graduating from Reed, you are working as a bartender in downtown Portland serving fancy cocktails, one of which features a large spherical ice cube as pictured below:

It seems reasonable that the ice is melting at a rate proportional to its surface area. Is the rate of change of the *radius* of the sphere increasing, decreasing, or remaining constant over time? These are the kinds of questions you are thinking about. (You'll need the following: the volume of a sphere is $V = \frac{4}{3}\pi r^3$ and the surface area of a sphere is $S = 4\pi r^2$ where r is the radius. To say that x is *proportional* to y means there is a constant α such that $x = \alpha y$.)

2. A town is located on the coast at the point labeled A below. Four miles down the beach and 2 miles out is an island, labeled B below. We would like to connect A and B with a cable (darkened below). To install the cable in the ocean costs twice as much as installing it along the beach.



Thus, for our purposes, using the variables given in the picture, we can take the cost to be

$$C = (4 - x) + 2y.$$

We would like to minimize the cost. This is an optimization problem, so let's apply our technique.

(a) Write C as a function of x.

(b) Identify the closed bounded interval to which x belongs for the purposes of our problem.

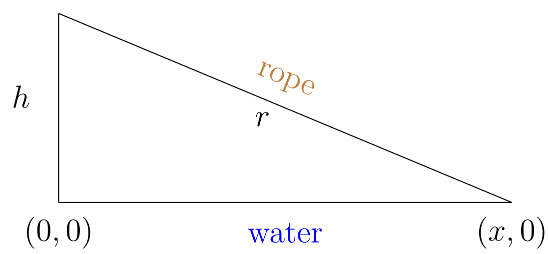
(c) The cost, C, is a continuous function of x. So confined to the closed interval you just identified, according to the extreme value theorem, it will have a minimum and a maximum, and they will occur either at an endpoint of the interval or at a point at which the derivative of C is 0. To get started, compute the derivative of C with respect to x. (Recall that the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$. You'll also need to use the chain rule.)

(d) Find any points x in your interval at which $C'(x) = 0$.

(e) Compute, using a calculator, the value of C at the endpoints of the interval and at the point where $C'(x) = 0$. Identify the minimum and the maximum values.

Week 8, Tuesday

1. Suppose you are standing on a dock pulling in a boat attached to a rope. The relevant picture is:



   The height of the point where the rope is being pulled is h feet above the water and does not change over time. The boat is x feet away from the dock, and the rope has length r.

   Suppose that we pull the rope in at a rate so that the speed at which the boat approaches the dock is a constant, $-k$ where $k > 0$. (We take the speed to be negative since the distance from the dock is decreasing.)

   (a) Write of formula for dr/dt, the rate the rope is being pulled in, as a function of r. Your formula should involve the speed of the boat and the length of the rope only (not x).

   (b) What does your formula tell you about dr/dt when (i) the boat is very far from the dock, and (ii) when the boat is very close to the dock?

2. Let $f(x) = (x + 2)^2(x - 1)$. Analyze f as we did in last Monday's lecture by completing the following:

   (a) Compute the critical points of f, i.e., those points where $f'(x) = 0$ or where $f'(x)$ fails to exist. Evaluate f at the points where $f'(x) = 0$.

   (b) Determine the sign of f' between the critical points (in order to figure out how the slope of f changes).

   (c) Find the *zeros* of f, i.e., those points where $f(x) = 0$, and evaluate f at these points.

   (d) What happens to $f(x)$ as $x \to \infty$ and as $x \to -\infty$? (Are there horizontal asymptotes?)

   (e) Find places where the functions "blows up", i.e., find any vertical asymptotes. If you find a vertical asymptote, how does f behave on either side of it?

(f) Graph f, labeling the points $(x, f(x))$ where $f(x) = 0$ (the zeros of f) and where $f'(x) = 0$ (the critical points).

3. Repeat the previous problem but for the function

$$f(x) = \frac{x+3}{x^2}.$$

(You should find two critical points.)

Week 8, Friday

1. Consider the function
$$S(x) = x + \frac{1}{x}$$
for $x > 0$.

   (a) Use the derivative of S to decide where S is decreasing and where S is increasing on $(0, \infty)$.

   (b) Does S have a minimum on $(0, \infty)$? If so, what is the minimal value, and where does it occur? Justify your solution in any case.

2. Find the point $(x, \sqrt{x})$ on the graph of $f(x) = \sqrt{x}$ that is closest to the point $(4, 0)$. **Justify your solution.** Recall that by the Pythagorean theorem, the distance between two points (a, b) and (c, d) in the plane is $d = \sqrt{(a-c)^2 + (b-d)^2}$. To make the calculation easier, note that the distance, d, is minimal exactly when d^2 is minimal. You can justify your solution by analyzing a derivative as in problem 1.

3. For each of the following sets X, answer the following questions:

   (i) Is X bounded above? If so: (i) what is its least upper bound, $\text{lub}(X)$, and (ii) is $\text{lub}(X)$ an element of X?

   (ii) Is X bounded below? If so: (i) what is its greatest lower bound, $\text{glb}(X)$, and (ii) is $\text{glb}(X)$ an element of X?

   (a) $X = \{23, \pi, 1\}$, as set of three real numbers.

   (b) $X = (-3, 10] = \{x \in \mathbb{R} : -3 < x \leq 10\}$, and interval.

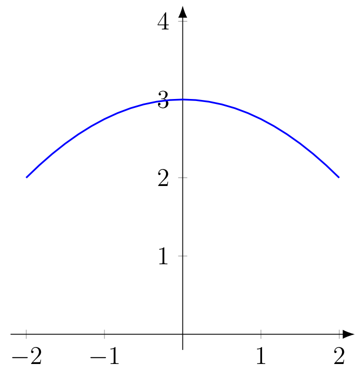
   (c) $X = \{-(1 + \frac{1}{1}), (1 + \frac{1}{2}), -(1 + \frac{1}{3}), (1 + \frac{1}{4}), -(1 + \frac{1}{5}), \dots\}$, an infinite sequence of real numbers.

Week 9, Tuesday

Consider the function

$$f(x) = -\frac{1}{4}x^2 + 3$$

on the interval $[-2, 2]$, as pictured below.



Graph of $f(x) = -\frac{1}{4}x^2 + 3$.

We are interested in approximating the area under f on this interval. Consider the partition of $[-2, 2]$ given by $P = \{-2, 1, 0, 1, 2\}$ with subintervals $[-2, -1]$, $[-1, 0]$, $[0, 1]$, and $[1, 2]$, each of length 1.

1. Upper sum:

   (a) Compute the least upper bounds for f on each subinterval:

   $$M_1 = \operatorname{lub} f([-2,-1]), \qquad M_2 = \operatorname{lub} f([-1,0]),$$
   $$M_3 = \operatorname{lub} f([0,1]), \qquad M_4 = \operatorname{lub} f([1,2]).$$

   Recall that, for instance, $f([-2, -1])$ is the set of numbers obtained by evaluating f at every number in $[-2, -1]$,

   $$f([-2,-1]) := \{f(x) : -2 \leq x \leq -1\}.$$

   It's the set of all "heights" of points on that portion of the graph of f sitting above the interval $[-2, -1]$. (Warning: the least upper bounds don't always occur at the right endpoints, and they don't always occur at the left endpoints. The same comment will apply the greatest lower bounds, below.)

   (b) Compute the upper sum $U(f, P)$ for f for this partition, an overestimate of the area.

(c) Draw a picture of the graph of f and of the rectangles whose areas appear as summands in $U(f,P)$.

2. Lower sum:

   (a) Compute m_1, m_2, m_3, and m_4, the greatest lower bounds for f on each subinterval:

$$\begin{array}{ll} m_1 = \operatorname{glb} f([-2,-1]), & m_2 = \operatorname{glb} f([-1,0]), \\ m_3 = \operatorname{glb} f([0,1]), & m_4 = \operatorname{glb} f([1,2]). \end{array}$$

   (b) Compute the lower sum $L(f,P)$ for f for this partition, an underestimate of the area.

   (c) Draw a picture of the graph of f and of the rectangles whose areas appear as summands in $L(f,P)$.

3. The actual area is $32/3 = 10.666\ldots$ Verify that

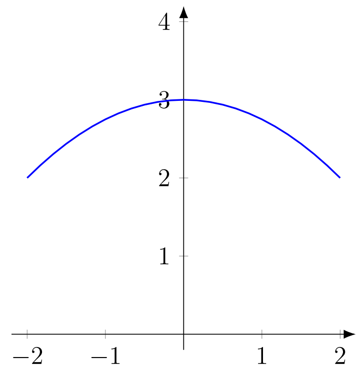
$$L(f,P) \leq \frac{32}{3} \leq U(f,P).$$

Week 9, Friday

1. Consider the function

$$f(x) = -\frac{1}{4}x^2 + 3$$

on the interval $[-2, 2]$, as pictured below.



Graph of $f(x) = -\frac{1}{4}x^2 + 3$.

(a) Let $P = \{2, -2\}$, be the partition of $[-2, 2]$ consisting of just two tick marks. This is in some sense the coarsest partition allowable. Compute the upper sum $U(f, P)$ and the lower sum $L(f, P)$ for this partition. (Show your work, of course.)

(b) Now consider the partition $Q = \{-2, -1/4, 1, 2\}$.

i. What are the subintervals for this partition?

ii. Using the notation from class, compute the M_i and m_i for these subintervals.

iii. Compute the upper and lower sums $U(f, Q)$ and $L(f, Q)$.

(c) Order the set of numbers

$$\left\{\frac{32}{3}, U(f, P), L(f, P), U(f, Q), L(f, Q)\right\}$$

from least to greatest. (The number 32/3 is the actual area under the graph. There is a general principle that says if you refine a partition by adding tick marks, the upper sums and lower sums get better, i.e., they get closer to the actual integral. This means the upper sums should get smaller and the lower sums should get larger when you refine a partition.)

(OVER)

(d) Find a function $g(x)$ whose derivative is $f(x) = -x^2/4+3$, and compute $g(2)-g(-2)$, the net change in g over the interval $[-2, 2]$.

2. Review problem: Find all points on the circle $x^2 + y^2 = 25$ where the slope is $3/4$.

Week 10, Tuesday

Let f be a function defined on an interval $[a, b]$, and let $P = \{t_0, t_1, \dots, t_n\}$ be a partition of $[a, b]$. To define a *Riemann sum* for f with respect to P, pick *any* points $c_i \in [t_{i-1}, t_i]$, one for each subinterval for P. The corresponding Riemann sum is

$$\begin{aligned} R(f, P) &:= f(c_1)(t_1 - t_0) + f(c_2)(t_2 - t_1) + \cdots + f(c_n)(t_n - t_{n-1}) \\ &= \sum_{i=1}^{n} f(c_i)(t_i - t_{i-1}). \end{aligned}$$

Since there are many choices for the c_i, there are many possible Riemann sums for the same partition.

1. Using the above notation, let

   $$M_i = \operatorname{lub} f([t_{i-1}, t_i]) \quad \text{and} \quad m_i = \operatorname{glb} f([t_{i-1}, t_i]),$$

   as usual. Your answers to the following should consist of complete sentences:

   (a) Explain why $m_i \le f(c_i) \le M_i$ for $i = 1, \dots n$.

   (b) Explain why $L(f, P) \le R(f, P) \le U(f, P)$.

2. Let $f(x) = x^2 + 1$ on the interval $[0, 3]$. Let $P = \{0, 1, 2, 3\}$ be a partition of $[0, 3]$.

   (a) Compute a Riemann sum $R(f, P)$ by choosing each c_i to be the midpoint of the i-th subinterval.

   (b) Compute $L(f, P)$ and $U(f, P)$, and check that

   $$L(f, P) \le R(f, P) \le U(f, P).$$

   (c) Compute the exact area under f on this interval by using the fundamental theorem of calculus to compute $\int_0^3 f(x)\, dx$.

3. Let $f(x) = 2x - 1$.

   (a) Draw the graph of f on the interval $[1, 3]$.

   (b) Compute the area under f using high school geometry (not calculus).

   (c) Use the fundamental theorem of calculus to compute the area under f from $x = 1$ to $x = 3$.

Week 10, Friday

1. Let $f(x) = x^2 - 4$.

   (a) Compute $\int_{-2}^{2} f(x)\,dx$.

   (b) Graph f on $[-2, 2]$.

   (c) In terms of the graph, explain what $\int_{-2}^{2} f(x)\,dx$ is measuring.

2. Suppose that $\int_0^3 g(x)\,dx = 4$, $\int_3^5 g(x)\,dx = 6$, and $\int_0^5 h(x)\,dx = -3$. Compute the following:

   (a) $\int_0^5 g(x)\,dx$.

   (b) $\int_0^5 (3\,g(x) + 5\,h(x))\,dx$.

3. You may find our essential derivatives handout helpful for this problem. (There is a link to it on our homepage.) Compute the following indefinite integrals (remember to add a constant c to each answer since the antiderivative is only determined up to the addition of a constant):

   (a) $\displaystyle\int (x + 3)\,dx$.

   (b) $\displaystyle\int (x^{3/2} + 2x + 1)\,dx$.

   (c) $\displaystyle\int \frac{1}{x^3}\,dx$.

   (d) $\displaystyle\int (☺ + 1)(2☺ - 3)\,d☺$.

   (e) $\displaystyle\int y^2\sqrt{y}\,dy$.

   (f) $\displaystyle\int dx$.

   (g) $\displaystyle\int (t^2 - \sin(t))\,dt$.

   (h) $\displaystyle\int (\sec^2(\theta) + 1)\,d\theta$.

   (i) $\displaystyle\int e^x\,dx$.

   (j) $\displaystyle\int \frac{1}{x}\,dx$.

Week 11, Tuesday

Please show your work for the following computations.

1. Compute the following definite integrals:

   (a) $\int_1^8 \sqrt{\frac{2}{x}}\,dx.$

   (b) $\int_{-\pi/2}^{\pi/2} (2t + \cos(t))\,dt.$

2. Compute the following indefinite integrals using u-substitutions. Be explicit about your substitution: $u =$, $du =$?. Remember to add $+c$ to get the most form for an antiderivative, and remember to express your solution in terms of the original variable (not u).

   (a) $\int 2x(x^2 - 9)^3\,dx.$

   (b) $\int t^2(t^3 + 5)^4\,dt.$

   (c) $\int \sin^5(3\theta)\cos(3\theta)\,d\theta.$

   (d) $\int -4x(1 - 2x^2)^{1/3}\,dx.$

   (e) $\int \frac{x^3}{(1 + x^4)^2}\,dx.$

   (f) $\int \frac{x^3}{\sqrt{1 + x^4}}\,dx.$

   (g) $\int (y + 1)\sqrt{2 - y}\,dy.$

3. Compute the following indefinite integrals by parts. Be explicit with your substitutions: $u =$?, $dv =$? (and remember to add $+c$).

   (a) $\int 3t\cos(2\pi t)\,dt.$

   (b) $\int (4 + 3x)e^x\,dx.$

Week 11, Friday

1. Let $f(x) = 4x^3 + 2x + 1$.
   (a) Compute $\int_1^x f(t)\,dt$.
   (b) Take the derivative of your solution to the first part of this problem, and show that it satisfies the statement of FTC2.

2. What is the derivative of the function

$$g(x) = \int_4^x \sec^5(t)\,dt$$

   with respect to x?

3. Define

$$h(x) = \int_1^{\sin(x)} \sqrt{\cos(t)}\,dt.$$

   We are interested in taking the derivative of $h(x)$. It is not a straightforward application of FTC2 since FCT2 involves integrals of the form $\int_a^x f(t)\,dt$, where the limit on top is x, not $\sin(x)$. We can evaluate this integral, though, using the chain rule. Define $k(x) = \int_1^x \sqrt{\cos(t)}\,dt$, and $\ell(x) = \sin(x)$.
   (a) Let f and g be arbitrary functions. What does the chain rule say about $f(g(x))$?
   (b) How is $h(x)$ related to $k(x)$ and $\ell(x)$?
   (c) Compute $h'(x)$ using the chain rule and FTC2.

4. We would like to prove that $\ln(xy) = \ln(x) + \ln(y)$ for all $x > 0$ and $y > 0$.
   (a) As a warm up, compute the derivatives of the following functions using the chain rule and the fact that $(\ln(x))' = 1/x$.
      i. $\ln(4x)$.
      ii. $\ln(x^2 + x + 3)$.
   (b) Now fix $y > 0$ and define a function of just the variable x by

$$f(x) = \ln(xy).$$

      i. Prove that $f'(x) = 1/x$ using the chain rule. (Recall that y is a constant.)
      ii. Since $f'(x) = (\ln(x))'$, these two functions must differ by some constant c:

$$f(x) = \ln(x) + c.$$

      Evaluate f at 1 to determine the value of c and establish the result we are trying to prove. (Recall the definition of f.)

Week 12, Tuesday

Please show your work for these problems.

1. Use properties of exponents and the natural logarithm to solve for x:
   (a) $\ln x = e$.
   (b) $e^{x^2-1} - 1 = 0$.
   (c) $27^x = \dfrac{9^{2x-1}}{3^{2x}}$.
   (d) $\dfrac{1}{e^{-\ln x}} = 5$.
   (e) $e^{\ln 2x} = 12$.

2. Take the derivatives of the following functions with respect to x:
   (a) $\ln(4x^2 + 3x + 1)$.
   (b) $\ln e^x$.
   (c) $\dfrac{1}{e^x + e^{-x}}$.
   (d) $xe^{-2/x}$.
   (e) $\ln \sqrt{x}$.

3. Compute the following indefinite integrals (remembering to add $+c$):
   (a) $\displaystyle\int \frac{6x^2}{x^3+5}\,dx$.
   (b) $\displaystyle\int \frac{e^x}{1+e^x}\,dx$.
   (c) $\displaystyle\int \ln(x)\,dx$ (hint: integration by parts).
   (d) $\displaystyle\int \frac{\ln x}{x}\,dx$.
   (e) $\displaystyle\int \frac{\sin(x)}{1+\cos(x)}\,dx$.

4. We know that $(e^x)' = e^x$ and $\ln'(x) = 1/x$. You can use these facts to find $(2^x)'$. Let $y = 2^x$. We want to compute y' (where the derivative is with respect to x).

Taking logs, we get $\ln y = \ln(2^x)$. By a property of the logarithm, we have $\ln 2^x = x\ln(2)$. Hence,

$$\ln y = x\ln(2).$$

Use implicit differentiation to compute y' and express your solution solely in terms of x (i.e., if a y appears in your solution, replace it by 2^x).

5. Compute the equation of the tangent line to the graph of $y = e^{-3x}$ at the point $(0, 1)$ in the form $y = mx + b$.

Week 13, Tuesday

1. Find the indefinite integrals (show your work):

   (a) $\int \sin(3x)e^{\cos(3x)}\,dx$

   (b) $\int x^2 e^{4x}\,dx.$

2. Let $y(t)$ be represent the size of a population varying over time. In class, we saw that if $y(t)$ satisfied the differential equation

$$y'(t) = r(S - y(t))$$

   for some positive constants r and S, then

$$y(t) = S - (S - I)e^{-rt}$$

   where $I = y(0)$, the initial population. Thus, over time the population converges exponentially quickly to S with the rate of convergence governed by the number r.

   Suppose instead that $y(t)$ satisfies

$$y'(t) = rt(S - y(t)). \tag{38.1}$$

   This model is similar to the first one but now the rate of convergence is governed by rt as opposed to r.

   (a) Find the most general solution $y(t)$ to equation (38.1).

   (b) What is the limit, $\lim_{t\to\infty} y(t)$?

   (c) Find the particular solution when the initial population is $2S$.

   (d) Find the particular solution when the initial population is $S/2$.

3. Suppose that swimming pool contains 13,000 gallons of water and 350 pounds of salt. As it rains over the winter, water is drained from the pool at a rate of 225 gallons per week to keep the water level constant. How much salt is in the pool after 7 months (28 weeks)?

   The difficulty with this question is that as the water is drained and replaced by rainwater, the salt concentration is decreasing. We can use a differential equation to solve the problem. Le $s(t)$ denote the number of pounds of salt in the pool at time t (with t measured in weeks).

(a) In terms of $s(t)$, what is the concentration of salt in the pool at time t (in pounds/gallon)?

(b) We have

$$s'(t) = -\text{ (number of pounds of salt lost per week)}.$$

Write a differential equation for $s'(t)$ in terms of $s(t)$.

(c) Solve the differential equation to find $s(t)$ with initial condition $s(0) = 350$ pounds.

(d) How many pounds of salt are lost after 28 weeks? Use a calculator to give an approximate solution. (This is the amount I will have to add at the beginning of the swimming season.)

Week 14, Tuesday

1. Prove that if $f(x) = 2x^2 + 4x + 1$, then $f'(x) = 4x + 4$, using the definition of the derivative.

2. Carefully state what it means to say $\lim_{x \to c} f(x) = L$. (Give the definition.)

3. Compute the equation of the tangent line to the graph of the function $f(x) = \ln(4x)$ at the point $x = 2$.

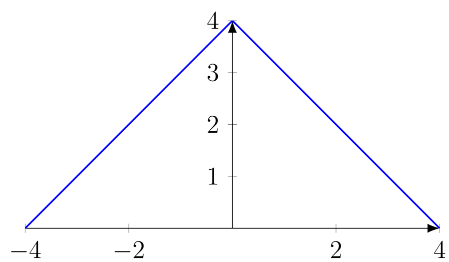
4. Each edge of a cube is expanding at a rate of 3 centimeters per second. How fast is the volume of the cube increasing when each edge is (i) 1 centimeter? (ii) 10 centimeters?

5. Let $f(x) = -x^2 + 3x$.

   (a) On the interval $[0, 3]$, what is the minimum value of f, and where does the minimum occur?

   (b) On the interval $[0, 3]$, what is the maximum value of f, and where does the maximum occur?

6. Consider the function $f(x) = 4 - |x|$ on the interval $[-4, 4]$:



Graph of $f(x) = 4 - |x|$.

   Compute the lower and uppers sums, $L(f, P)$ and $U(f, P)$ with respect to the partition $P = \{-4, -2, 0, 2, 4\}$.

7. Compute the following integrals.

   (a) $\int_0^2 (3x-1)^2\,dx$. (Note: this is a definite integral. There rest of the integrals in this problem are indefinite.)

   (b) $\int x^2 \sqrt{x+2}\,dx$.

(c) $\int \sin(x)\cos(x)\,dx.$

(d) $\int \frac{x}{(x+1)^2}\,dx.$

(e) $\int x^2 \ln(x)\,dx.$

8. Let $P(t)$ represent a population of some sort. Suppose P satisfies the differential equation

$$P'(t) = P(t)\sin(t).$$

(a) Find the most general solution to the equation.

(b) Find the solution that satisfies $y(0) = 10$.

(c) Describe the general behavior of the population over time.

HANDOUTS

Essential derivatives

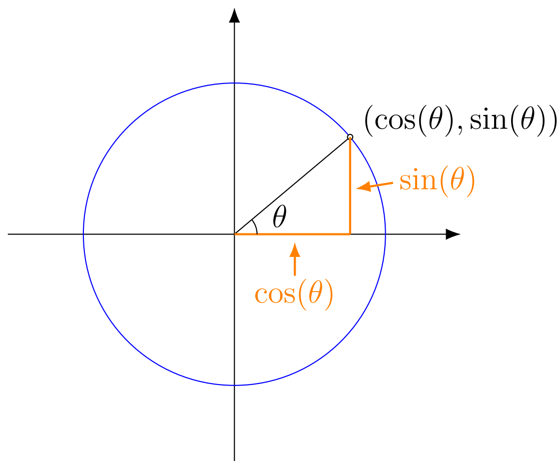
- $\frac{d}{dx}\, c = 0$ for each constant c
- $\frac{d}{dx}\, x^\alpha = \alpha\, x^{\alpha-1}$ for every real number α
- $\frac{d}{dx}\, e^x = e^x$
- $\frac{d}{dx}\, \ln\, x = \frac{1}{x}$
- $\frac{d}{dx}\, \cos x = -\sin x$
- $\frac{d}{dx}\, \sin x = \cos x$
- $\frac{d}{dx}\, \tan x = \sec^2 x$
- $\frac{d}{dx}\, \cot x = -\csc^2 x$
- $\frac{d}{dx}\, \sec x = \sec x \tan x$
- $\frac{d}{dx}\, \csc x = -\csc x \cot x$

- $\frac{d}{dx}\, b^x = b^x \ln\, b$ for all positive constants b
- $\frac{d}{dx}\, \arccos x = -\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\, \arcsin x = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\, \arctan x = \frac{1}{1+x^2}$

- $(f+g)' = f' + g'$ and $(cf)' = c(f')$ [linearity]
- $(fg)' = f'\,g + f\,g'$ [product or Leibniz rule]
- $\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$ [quotient rule]
- $(f \circ g)'(x) = f'(g(x))g'(x)$ [chain rule]

Trigonometry review

Definition. $(\cos(\theta), \sin(\theta))$ is the point on the unit circle centered at the origin determined by the angle θ as in the following picture:



The tangent, cotangent, secant, and cosecant functions are defined in terms of sine and cosine:

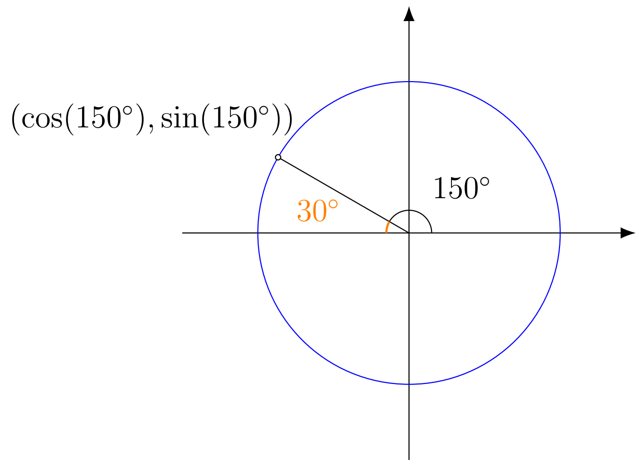
$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \quad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}, \quad \sec(\theta) = \frac{1}{\cos(\theta)}, \quad \csc(\theta) = \frac{1}{\sin(\theta)},$$

Standard angles. The values of cosine and sine at the standard angles:

θ	$\cos(\theta)$	$\sin(\theta)$
0	1	0
$30^\circ = \frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$45^\circ = \frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$60^\circ = \frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$90^\circ = \frac{\pi}{2}$	0	1

From these angles in the first quadrant, you can derive values for cosine and sine for corresponding angles in the other quadrants.

Examples. The angle 150° is 30° up from 180° as shown below:



Therefore,

$$(\cos(150°), \sin(150°)) = (-\cos(30°), \sin(30)) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

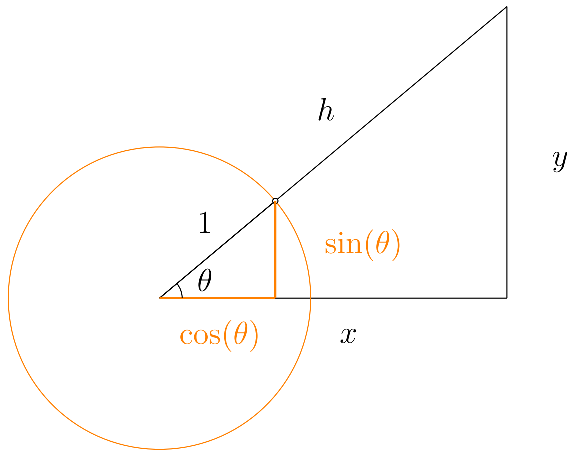
Converting between radians and degrees. If

$$\theta = x \text{ radians} = y \text{ degrees},$$

then "x is to π as y is to 180":

$$\frac{x}{\pi} = \frac{y}{180}.$$

Similar triangles.



$$\frac{\text{adj}}{\text{hyp}} = \frac{x}{h} = \frac{\cos(\theta)}{1} = \cos(\theta)$$

$$\frac{\text{adj}}{\text{hyp}} = \frac{y}{h} = \frac{\sin(\theta)}{1} = \sin(\theta)$$

$$\frac{\text{opp}}{\text{adj}} = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$$

$$\frac{\text{adj}}{\text{opp}} = \frac{x}{y} = \frac{\cos(\theta)}{\sin(\theta)} = \cot(\theta)$$

$$\frac{\text{hyp}}{\text{adj}} = \frac{h}{x} = \frac{1}{\cos(\theta)} = \sec(\theta)$$

$$\frac{\text{hyp}}{\text{opp}} = \frac{h}{y} = \frac{1}{\sin(\theta)} = \csc(\theta)$$

Identities. Since $(\cos(\theta), \sin(\theta))$ is a point on the unit circle,

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

The **sum formulas**:

$$\begin{aligned} \cos(\theta + \psi) &= \cos(\theta)\cos(\psi) - \sin(\theta)\sin(\psi) \\ \sin(\theta + \psi) &= \sin(\theta)\cos(\psi) + \sin(\theta)\cos(\psi). \end{aligned}$$

Letting $\theta = \psi$ in the sum formulas gives the **double-angle** formulas:

$$\begin{aligned} \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ \sin(2\theta) &= 2\sin(\theta)\cos(\theta). \end{aligned}$$

Several identities follow from the double-angle formula for cosine. First substitute $\cos^2(\theta) = 1 - \sin^2(\theta)$ to get

$$\cos(2\theta) = 1 - 2\sin^2(\theta).$$

Or, substitute $\sin^2(\theta) = 1 - \cos^2(\theta)$ to get

$$\cos(2\theta) = 2\cos^2(\theta) - 1.$$

Solve for $\sin^2(\theta)$ and for $\cos^2(\theta)$ in these last two equations, respectively, to get the **half-angle** formulas:

$$\begin{aligned} \sin^2(\theta) &= \frac{1 - \cos(2\theta)}{2} \\ \cos^2(\theta) &= \frac{1 + \cos(2\theta)}{2}. \end{aligned}$$

Limits. The following is proved in Math 112:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0.$$

Definition of the integral

Let f be a bounded function on a closed interval $[a, b]$. (The word "bounded" here means that there is some constant B such that $-B \leq f(x) \leq B$ for all $x \in [a, b]$.) A *partition* of $[a, b]$ is a set $P = \{t_0, t_1, t_2, \ldots, t_n\}$ with each $t_i \in [a, b]$ and with $t_0 = a$ and $t_n = b$. By convention, we always take $t_0 < t_1 < \cdots < t_n$. In that case the interval $[t_{i-1}, t_i]$ is called the *i-th subinterval of P*. The values the function f takes on the i-th interval is denoted $f([t_{i-1}, t_i])$:

$$f([t_{i-1}, t_i]) = \{f(x) : t_{i-1} \leq x \leq t_i\}.$$

This set is called the *image of* $[t_{i-1}, t_i]$ under f. For each $i = 1, \ldots, n$ define

$$\begin{aligned} m_i &= \operatorname{glb} f([t_{i-1}, t_i]), \\ M_i &= \operatorname{lub} f([t_{i-1}, t_i]), \end{aligned}$$

then define the *lower and upper sums for f with respect to P*:

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{n} m_i (t_i - t_{i-1}), \\ U(f, P) &= \sum_{i=1}^{n} M_i (t_i - t_{i-1}). \end{aligned}$$

The *upper and lower integrals* for f are:

$$\begin{aligned} L\int_a^b f &= \operatorname{lub} \{L(f, P) : P \text{ a partition of } [a, b]\} \\ U\int_a^b f &= \operatorname{glb} \{U(f, P) : P \text{ a partition of } [a, b]\}. \end{aligned}$$

The function f is *integrable* if

$$L\int_a^b f = U\int_a^b f,$$

and in that case, the common value is the *integral of f on* $[a, b]$, denoted $\int_a^b f$ or $\int_a^b f(x)\,dx$. □

Riemann sum. Pick real numbers c_i such that $t_{i-1} \leq c_i \leq t_i$ for $i = 1, \ldots, n$. The quantity

$$R = \sum_{i=1}^{n} f(c_i)(t_i - t_{i-1})$$

is called a *Riemann sum* for f relative to the partition P. Note that

$$L(f, P) \leq R \leq U(f, P)$$

for all choices of the c_i. Thus, a Riemann sum estimates the integral of f, provided the integral exists.