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"Equivariant algebraic K-theory"

5-30-15

Rings $\circlearrowleft G$

Schemes $\xrightarrow{K_i}$ Ab / Mackey Functors

Spaces

$\searrow$ ∞-loop spaces / Conn Ω-spectra $\circlearrowleft G$ $\xrightarrow{\pi_i}$ Ab

$$KR = \Omega^n X_n$$

want: $\underline{K_G R \simeq \Omega^V X_V \quad V \text{ rep}}$

$R \circlearrowleft G \quad \rightsquigarrow \quad$ get $\quad Mod(R) \circlearrowleft G$

Let Mod be cat of f.g. modules

$$M = M \quad \text{as ab group}$$

$$R \times M \xrightarrow{g \times id} R \times M \xrightarrow{\gamma} M$$

Another problem: $\quad R \circlearrowleft G \to S \circlearrowleft G \qquad$ G-map

$$Mod(R) \circlearrowleft G \xrightarrow[- \otimes_R S]{} Mod(S) \circlearrowleft G$$

not a G-map. exercise: $g M \otimes_R S \neq g(M \otimes_R S)$

• can define as isu

$$g M \otimes_R S \xrightarrow{\cong} g(M \otimes_R S)$$

<u>Recall</u>: KR = gp completion of $B(\text{iso}\, P(R))$

f.g. proj R-mod

to get spectrum:

iso $P(R)$
(symmetric monoidal cat)
$\rightsquigarrow$

May - operadic
Segal - (Γ-space)

infinite loop space machines

$\rightsquigarrow$ Ω-spectrum w/ 0 space KR

<u>Thm</u> (May–Thomason) These machines agree

Equivariantly: genuine symm mon. $\mathcal{C} \circlearrowleft G$ $\rightsquigarrow$

operadic - Guillou-May
Segal - Shimakawa

$\longrightarrow$ genuine Ω-G-spectrum = $RO(G)$-graded spectrum

0 space = group completion $B\mathcal{C}$

G finite <u>Thm</u>: (May M Osorno) These machines agree

What is "genuine"?

- transfers? $\mathcal{C}^H \longrightarrow \mathcal{C}^G$

$$\prod_{G/H} \mathcal{C} \longrightarrow \mathcal{C}$$

$$c_1, \dots, c_j \longmapsto c_1 \oplus \dots \oplus c_j$$

if this were G equiv, done

__not__ only G-equiv up to isomorphism

__Intuition__: __Recall__ 0^{th} component of KR = gp completion of $\underbrace{\coprod_n BGL_n(R)}_{\text{Classifying spaces of } GL_n(R)\text{-bundles}}$

__idea__: replace by equiv $GL_n(R) \rtimes G$ bundles

$$\begin{array}{c} \Pi \circlearrowright E \circlearrowleft G \\ \downarrow \\ B \end{array}$$

__Thm__ (Guillou, May, M) G finite, discrete

Π compact Lie

$B\mathcal{C}at(\tilde{G}, \tilde{\Pi}) \to B\mathcal{C}at(\tilde{G}, \Pi)$ is a univ princ $\Pi \rtimes G$ bundle

__Not'n__: $\tilde{G}$ = translation cat, i.e., $B\tilde{G} = EG$

$\mathcal{C}at(\mathcal{C}, \mathcal{D})$ = all functors and all nat trans. G acts by conj.

__Def__: $\mathcal{C}at(\tilde{G}, \mathcal{B})^G := \mathcal{C}^{hG}$ = htpy fixed pts of $\mathcal{C}$

<u>Guess:</u> $K_G(R) =$ gp compl of $\coprod B\,\mathcal{C}at(\widetilde{G}, GL_n(R))$

Is this an ∞-loop G-space?

E_∞-operads

	top	cat
non equiv	$\mathcal{O}(j) = E\Sigma_j$	$B\mathcal{O}(j) = E\Sigma_j$ $\mathcal{O}(j) = \widetilde{\Sigma}_j$
equiv	$\mathcal{O}_G(j) =$ univ $G \times \Sigma_j$-bundle	$\mathcal{O}_G(j) = \mathcal{C}at(\widetilde{G}, \widetilde{\Sigma}_j)$ (def Guillou-May

Thm (May): Permutative cats $\simeq$ alg / $\mathcal{O}$

Fact: Symm cat $\simeq$ pseudo alg / $\mathcal{O}$

<u>Def:</u> (Guillou-May) genuine perm G-cat alg/$\mathcal{O}_G$

——— symm G-cat pseudo-alg/$\mathcal{O}_G$

Examples: $\mathcal{C}$ permutative cat w/ G-action

$$\mathcal{O}(j) \times \mathcal{C}^j \to \mathcal{C}$$

apply $\mathcal{C}at(\tilde{G}, -)$: $\mathcal{C}at(\tilde{G}, \tilde{\Sigma}_j) \times \mathcal{C}at(\tilde{G}, \mathcal{C})^j \to \mathcal{C}at(G, \mathcal{C})$

$\Rightarrow \mathcal{C}at(\tilde{G}, \mathcal{C})$ is a genuine perm G-cat

Def: $K_G R =$ gp compl of $B\mathcal{C}at(\tilde{G}, \mathrm{iso}\, \mathcal{P}(R))$

agree with guess?

$\coprod_n GL_n R \xhookrightarrow[\text{skeleton}]{} \mathrm{iso}\, \mathcal{F}(R)$ not a G-map

Thm (M) $\mathcal{C}at(\tilde{G}, -)$ is wonderful:

- "htpy invariant" $\mathcal{C}^{?G} \xrightarrow[\simeq]{} \mathcal{D}^{?G}$ G-map

gives equiv of cat $\mathcal{C}^{hG} \to \mathcal{D}^{hG}$ (define $\coprod_n GL_n R \xleftarrow[G\text{-map}]{} \mathrm{iso}\, \mathcal{F}(R)$)

- $\mathcal{C}^{?G} \longrightarrow \mathcal{D}^{?G}$ pseudo-equivariant

$$\begin{array}{c} G \xrightarrow{\mathcal{C}} \mathcal{C}at \\ \Downarrow \text{ pseudo nat'l trans} \\ G \xrightarrow{\mathcal{D}} \mathcal{C}at \end{array}$$

then we can construct an on the nose equiv map

$$\mathcal{C}at(\tilde{G}, \mathcal{C}) \longrightarrow \mathcal{C}at(\tilde{G}, \mathcal{D})$$

<u>Thm</u> Properties of K_G

- $R \to K_G(R)$ is a functor
- $K_G(R)^H \simeq K(R_H[H])$ if $|H|^{-1} \in R$
- $K_G\left(\begin{array}{c}\mathbb{C}^{top} \circlearrowleft G \text{ trivial} \\ \mathbb{R}^{top} \circlearrowleft G \text{ trivial}\end{array}\right) \begin{array}{c}\simeq KU_G \\ \simeq KO_G\end{array}$
- $K_G(\mathbb{C}^{top} \circlearrowleft \mathbb{Z}/2 - \text{conj}) \simeq K_r$ Atiyah
- K_G is invariant under equiv. Morita equivalence
- For a Galois ext of rings S/R

  $K_G(S)^G \simeq KR$

- the map from QL conj

$$\begin{array}{ccc} KF & & \\ \parallel & & \\ KE^G & \to & KE^{hG} \\ \wr\! \| & & \wr\! \| \\ K_G E^G & \to & K_G E^{hG} \end{array}$$

E/F is Galois w/ gp G

- the rep assembly map

$$K\, Rep_F G \xrightarrow{\otimes_F E} K\, \underbrace{Mod_G(E)}_{E \text{ mod w/ semi lin } G\text{-action}}$$

is the fixed pt map of a G-map

$$(K_G F)^G \longrightarrow (K_G E)^G$$

<u>Equiv A-thy</u> $X = G$-space $R(X) \circlearrowleft G$

$\uparrow$ cat of retractive spaces

$X \xrightarrow{i} Y \xrightarrow{r} X$

$$g(X \xrightarrow{i} Y \xrightarrow{r} X) =$$

$$X \xrightarrow{g^{-1}} X \xrightarrow{i} Y \xrightarrow{r} X \xrightarrow{g} X$$

$R(X)^{hG}$ = retractive G-spaces i,r equiv + G-maps

<u>Def</u>: $A_G(X) = \Omega\, h_o S_\bullet\, \mathrm{Cat}(\widetilde{G}, R(X))$

<u>Prop</u> (Makiewich)

$$A_G(X)^H = A(EH \times_H X \to BH)$$

defined by B Williams

$$A_G(*)^H = A\, BH$$