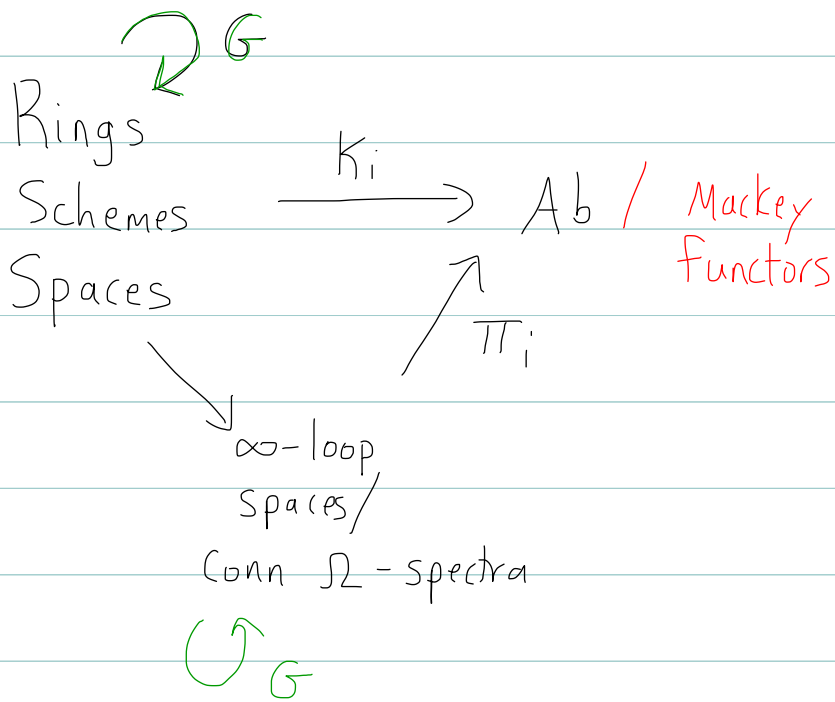


Mona Merling

"Equivariant algebraic
K-theory"

5-30-15



$$KR = \Omega^n X_n$$

want: $K_G R \cong \Omega^V X_V$ V rep

$R \curvearrowright G \rightsquigarrow$ get $\text{Mod}(R) \curvearrowright G$

$M = M$ as ab group

$$R \times M \xrightarrow{g \times \text{id}} R \times M \xrightarrow{g} M$$

Let Mod be
cat of f.g.
modules

Another problem: $\begin{matrix} \curvearrowright G & & \curvearrowright G \\ R & \rightarrow & S \end{matrix}$ G -map

$$\text{Mod}(R) \curvearrowright G \xrightarrow{- \otimes_R S} \text{Mod}(S) \curvearrowright G$$

not a G -map. exercise: $g M \otimes_R S \neq g(M \otimes_R S)$
• can define as iso

$$gM \otimes_R S \xrightarrow{\cong} g(M \otimes_R S)$$

Recall: $KR = gp$ completion of $B(\text{iso } P(R))$
 ↑ f.g. proj R -mod

to get spectrum:

iso $P(R)$
 (symmetric monoidal cat)



May-operadic
 Segal-(P-space)

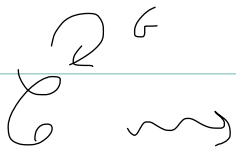
infinite loop space
 machines

→ Ω -spectrum w/ 0 space KR

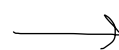
Thm (May-Thomason) These machines agree

Equivariantly:

genuine
 symm mon.



operadic-Guillon-May
 Segal-Shimakawa



genuine
 Ω - G -spectrum

0 space =
 group completion
 $B\mathcal{C}$

||
 $R_0(G)$ -graded
 Spectrum

G finite Thm: (May-Moskowitz) These machines agree

What is "genuine"?

• transfers?

$$\mathcal{C}^H \longrightarrow \mathcal{C}^G$$

$$\prod_n \mathcal{C} \longrightarrow \mathcal{C}$$

$$G/H \quad C_1, \dots, C_j \longmapsto C_1 \oplus \dots \oplus C_j$$

if this were G equiv, done

not only G -equiv up to isomorphism

Intuition: Recall 0^{th} component of $KR = \text{gp completion of}$

$$\coprod_n BGL_n(\mathbb{R})$$

idea: replace by equiv $GL_n(\mathbb{R}) \rtimes G$ bundles

Classifying spaces of

$GL_n(\mathbb{R})$ -bundles

$$\prod \mathcal{C} \supseteq E \supseteq G$$

$$\downarrow$$

Thm (Guillon, May, M) G finite, discrete
 \prod compact Lie

Not'n: $\tilde{G} = \text{translation}$

$B\text{Cat}(\tilde{G}, \tilde{\pi}) \rightarrow B\text{Cat}(\bar{G}, \pi)$ is a
 univ princ $\prod \rtimes G$ bundle

cat, i.e., $B\tilde{G} = EG$

$\text{Cat}(\mathcal{C}, \mathcal{D}) = \text{all functors and all nat trans. } G \text{ acts by conj.}$

Def: $\text{Cat}(\tilde{G}, B)^G = \mathcal{C}^{RG} = \text{htpy fixed pts of } \mathcal{C}$

Guess: $K_G(R) = \text{gp compl of } \coprod \text{Bcat}(\tilde{G}, GL_n(R))$

Is this an ∞ -loop G -space?

E_∞ -operads

| | top | cat |
|-----------|--|---|
| non equiv | $\mathcal{O}(j) = E\Sigma_j$ | $\text{B}\mathcal{O}(j) = E\Sigma_j$ $\mathcal{O}(j) = \tilde{\Sigma}_j$ |
| equiv | $\mathcal{O}_G(j) = \text{univ } G \times \Sigma_j$ -bundle | $\mathcal{O}_G(j) = \text{Cat}(\tilde{G}, \tilde{\Sigma}_j)$ (def Guillou-May) |

Thm (May): Permutative cats $\simeq \text{alg} / \mathcal{O}$

Fact: Symm cat $\simeq \text{pseudo alg} / \mathcal{O}$

Def: (Guillou-May)

| | |
|-----------------------|-------------------------------------|
| genuine perm G -cat | $\text{alg} / \mathcal{O}_G$ |
| — symm G -cat | $\text{pseudo-alg} / \mathcal{O}_G$ |

Examples: \mathcal{C} permutative cat w/ G -action

$$\mathcal{O}(j) \times \mathcal{C}^j \rightarrow \mathcal{C}$$

apply $\text{Cat}(\tilde{G}, -): \text{Cat}(\tilde{G}, \tilde{\Sigma}_j) \times \text{Cat}(\tilde{G}, \mathcal{C})^j \rightarrow \text{Cat}(\tilde{G}, \mathcal{C})$

$\Rightarrow \text{Cat}(\tilde{G}, \mathcal{C})$ is a genuine perm G -cat

Def: $K_G R = \text{gp compl of } \mathcal{B}\text{Cat}(\tilde{G}, \text{iso } \mathcal{P}(R))$

agree with guess?

$$\coprod_n GL_n R \xrightarrow[\text{skeleton}]{} \text{iso } \mathcal{F}(R) \quad \text{not a } G\text{-map}$$

Thm (M) $\text{Cat}(\tilde{G}, -)$ is wonderful:

• "htpy invariant" $\mathcal{C}^{\mathbb{Z}G} \xrightarrow{\cong} \mathcal{D}^{\mathbb{Z}G} \quad G\text{-map}$

gives equiv of cat $\mathcal{C}^{hG} \rightarrow \mathcal{D}^{hG} \quad \left(\begin{array}{l} \text{define} \\ \coprod_n GL_n R \xrightarrow[\text{G-map}]{} \text{iso } \mathcal{F}(R) \end{array} \right)$

• $\mathcal{C}^{\mathbb{Z}G} \rightarrow \mathcal{D}^{\mathbb{Z}G}$ pseudo-equivariant

$$\begin{array}{ccc}
 G & \xrightarrow{\mathcal{C}} & \text{Cat} \\
 & \downarrow \text{pseudo nat'l trans} & \\
 G & \xrightarrow{\mathcal{D}} & \text{Cat}
 \end{array}$$

then we can construct an on the nose equiv map

$$\text{Cat}(\tilde{G}, \mathcal{C}) \rightarrow \text{Cat}(\tilde{G}, \mathcal{D})$$

Thm Properties of K_G

- $R \rightarrow K_G(R)$ is a functor
- $K_G(R)^H \simeq K(R_H[M])$ if $|M|^{-1} \in R$
- $K_G(\mathbb{C}^{\text{top}} \curvearrowright G \text{ trivial}) \simeq KU_G$
 $\mathbb{R}^{\text{top}} \curvearrowright G \text{ trivial} \simeq KO_G$
- $K_G(\mathbb{C}^{\text{top}} \curvearrowright \mathbb{Z}/2\text{-conj}) \simeq KR$ Atiyah
- K_G is invariant under equiv. Morita equivalence
- For a Galois ext of rings S/R
 $K_G(S)^G \simeq KR$

• the map from QL con; $\begin{array}{ccc} \overset{KF}{\downarrow} & & \\ K E^G & \xrightarrow{\quad} & K E^{hG} \\ \downarrow & & \downarrow \\ K_G E^G & \xrightarrow{\quad} & K_G E^{hG} \end{array}$ E/F is Galois w/ gp G

• the rep assembly map

$$K \text{ Rep}_F G \xrightarrow{\otimes_F E} \underbrace{K \text{ Mod}_G(E)}_{E \text{ mod w/ semi lin } G\text{-action}}$$

is the fixed pt map of a G -map

$$(K_G F)^G \longrightarrow (K_G E)^G$$

Equiv A-thy $X = G$ -space $R(X) \cong^G$
 \uparrow
 cat of retractive spaces
 $X \xrightarrow{i} Y \xrightarrow{r} X$

$$g(x \xrightarrow{i} Y \xrightarrow{r} X) =$$

$$x \xrightarrow{g^{-1}} X \xrightarrow{i} Y \xrightarrow{r} X \xrightarrow{g} X$$

$R(X)^{hG} = \text{retractiv e } G\text{-spaces } i, r \text{ equiv } + G\text{-maps}$

Def: $A_G(X) = \Omega \text{ ho } S. \text{ Cat}(\tilde{G}, R(X))$

Prop (Makiewicz)

$A_G(X)^H = A(EH \times_H X \rightarrow BH)$ ← defined by
B Williams

$A_G(*)^H = \forall BH$