# Equivariant K-theory of Compact Lie Groups w/ Involution

Note Title 5/31/2015

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$G$ = compact simply. conn. Lie group

Brylinski-Zhang: computed $K_G^*(G)$

we: $K_{G \rtimes \mathbb{Z}/2}(G)$ $G$ have an involution $\alpha$

$T$ = max. torus

$W$ = Weyl gp.

$R(G)$ = complex rep. ring

$u_1, \cdots u_n$ = fundamentals weights

$\bar{u}_1, \cdots \bar{u}_n = \sum$ elts. in Weyl orbit of $u_i$

$$R(G) \subset R(T)$$

$$R(G) = \mathbb{Z}[\bar{u}_1, \cdots \bar{u}_n] \qquad R(T) = \mathbb{Z}[u_1, u_1^{-1}, \cdots, u_n, u_n^{-1}]$$

for any $G$: $K_G$ = $G$-equiv. K-theory

$$K_G(S^0) = R(G)$$

$$K_G(X) = R(G)\text{-module}$$

By Kunneth:

$$K_T(T) = \Omega_{R(T)/\mathbb{Z}} = R(T) \otimes \Lambda[du_1, \dots du_n]$$

Kahler differentials of $R(T)$ over $\mathbb{Z}$

<u>Recall:</u> $R \hookrightarrow S$ comm. rings

$\Omega_{S/R}$ is generated by $\{ds \mid s \in S\}$ / universal derivation rel.

$$dr = 0 \quad , r \in R$$
$$d(s_1 + s_2) = ds_1 + ds_2$$
$$d(s_1 s_2) = (ds_1) s_2 + s_1 ds_2$$

Th (Brylinski-Zhang) :

$$\begin{tikzcd}
K_G(G) \arrow[r, "\text{rest.}"] \arrow[d, "\cong"] & K_T(T) \arrow[d, "\cong"] \\
\Omega_{R(G)/\mathbb{Z}} \arrow[r] \arrow[d, "\cong"] & \Omega_{R(T)/\mathbb{Z}} \arrow[d, "\cong"] \\
\mathbb{Z}[\bar{u}_1, \cdots \bar{u}_n] \otimes \Lambda[d\bar{u}_1, \cdots d\bar{u}_n] \arrow[r, hook] & \mathbb{Z}[u_1, u_1^{-1}, \cdots u_n, u_n^{-1}] \otimes \Lambda[du_1, \cdots du_n]
\end{tikzcd}$$

$W \in R(G)$, $dW \in \Omega_{R(G)/\mathbb{Z}}$

$dW$ corresp. to a complex of $G$-vec. bundles

$$G \times \mathbb{R} \times W \longrightarrow G \times \mathbb{R} \times W$$

$$(g, t, w) \longmapsto \begin{cases} (g, t, w) & \text{if } t < 0 \\ (g, t, -t g(w)) & \text{if } t \geq 0 \end{cases}$$

<u>Our goal</u>: $K_{G \rtimes \mathbb{Z}/2}(G)$ if $G$ has involution $\alpha$

can form $G \rtimes \mathbb{Z}/2$ two ways:

1. $\mathbb{Z}/2$ acts on $G$ by $\alpha$
2. $\mathbb{Z}/2$ acts on $G$ by $\gamma : g \longmapsto \alpha(g)^{-1}$

$$\mathbb{Z}/2_+ \longrightarrow S^0 \longrightarrow S^A \leftarrow \text{real sign rep. of } \mathbb{Z}/2$$

$\alpha$ act on $R(G)$: $\alpha^*(\bar{u}_i) = \bar{u}_{\sigma(i)}$, $\sigma \in \Sigma_n$

let $\succ$ be any lin. ordering on subsets of $\{1, \dots n\}$

$$I_\sigma = \{\{i_1 < \dots < i_k\} \mid \{\sigma(i_1), \dots \sigma(i_k)\} \succ \{i_1, \dots i_k\}\}$$

$$J_\sigma = \{\{i_1 < \dots < i_k\} \mid \{\sigma(i_1), \dots \sigma(i_k)\} = \{i_1, \dots i_k\}\}$$

if $R \in J_\sigma$:

1. $\mathbb{Z}/2$ acts by $\alpha$: $\mathrm{orb}(R) = \#$ of free orbits in $R$
2. $\mathbb{Z}/2$ acts by $\gamma$: $\mathrm{orb}(R) = \#$ of all orbits in $R$

if $\varepsilon$ even : $S^{(\varepsilon)} = S^0$

if $\varepsilon$ odd : $S^{(\varepsilon)} = S^{A-1}$

Theorem (H., Kriz, Somberg) : as $R(G \rtimes \mathbb{Z}/2)$-modules

$$K_{G \rtimes \mathbb{Z}/2}(G) \cong$$

$S^0$ or $S^{1+A}$

$$K_{G \rtimes \mathbb{Z}/2}\Big( \bigvee_{I_\sigma} \Sigma^k \mathbb{Z}/2_+ \vee \bigvee_{J_\sigma} \Sigma^k S^{(orb\{i_1, \ldots i_k\})} \Big) \longleftarrow \star$$

What are these pieces?

$$K^*_{G \rtimes \mathbb{Z}/2}(S^0) = R(G \rtimes \mathbb{Z}/2) \quad \text{in dim } 0$$

$$K^*_{G\rtimes\mathbb{Z}/2}(\mathbb{Z}/2_+) = K^*_G(S^0) = R(G) \text{ in dim } 0$$

$$\mathbb{Z}/2_+ \longrightarrow S^0 \longrightarrow S^A$$

$$0 \to K^0_{G\rtimes\mathbb{Z}/2}(S^A) \to R(G\rtimes\mathbb{Z}/2) \longrightarrow R(G) \to K^1_{G\rtimes\mathbb{Z}/2}(S^A) \to 0$$

$K^0_{G\rtimes\mathbb{Z}/2}(S^A)$ || free. ab. on irred G-rep. do extend

$K^1_{G\rtimes\mathbb{Z}/2}(S^A)$ || free ab. on irred. G-rep. do not extend to $G\rtimes\mathbb{Z}/2$

Rough Idea of Proof: construct map from ☆ to

$$F(G_+, K_{G \rtimes \mathbb{Z}/2}) \leftarrow F(G_+, K_{G \rtimes \mathbb{Z}/2})^G_* = K_G(G)$$

show it is equiv. on $(G \rtimes \mathbb{Z}/2)$-fixed pts.

for $I_\sigma$: $d\bar{u}_{i_1} \wedge \cdots d\bar{u}_{i_k} : S^k \to F(G_+, K_{G \rtimes \mathbb{Z}/2})$

$$\underbrace{(G \rtimes \mathbb{Z}/2) \wedge_G S^k}_{\Sigma^k \mathbb{Z}/2_+} \xrightarrow{G\text{-map}} F(G_+, K_{G \rtimes \mathbb{Z}/2})$$

$(G \rtimes \mathbb{Z}/2)$-map

$\alpha^*: R(G)$

for $J_\sigma$: single orbit has $k=1$ or $2$

if $k=1$. $\quad \alpha^*(\overline{u}_{i_1}) = \overline{u}_{i_1}$

$$d\overline{u}_{i_1} : S^1 \longrightarrow F(G_+, K_{G\rtimes \mathbb{Z}/2})$$

lifts $\quad K^{1 \text{ or } A}_{G\rtimes \mathbb{Z}/2}(G)$

if $k=2$. $\quad d\overline{u}_{i_1}, d\overline{u}_{i_2}$:

$$d\overline{u}_{i_1} : S^1 \longrightarrow F(G_+, K_{G\rtimes \mathbb{Z}/2}) \quad G\text{-map}$$

multiplicative norm:

$$N_G^{G\rtimes \mathbb{Z}/2}(d\overline{u}_{i_1}) : S^{1+A} \longrightarrow F(G_+, K_{G\rtimes \mathbb{Z}/2})$$

$$\left( N_H^G(X) := \bigwedge_{G/H} X \right)$$

__Examples:__

note: if $\alpha$ inner, ~~k~~ $\alpha^*$ trivial

symmetric spaces $G/G^\alpha$

__Example 1:__ $G = SU(2)$, $G/G^\alpha = SU(2)/SO(2)$,

$\mathbb{Z}/2$ acts by $\gamma$

$\overline{u}_1$

cx. sign. rep

$$R(SU(2)) = \mathbb{Z}[z]$$

$$R(SU(2) \rtimes \mathbb{Z}/2) = \mathbb{Z}[z, q] / q^2 - 1$$

wedge $= S^0 \vee S^A$

$$0 \to K^0_{G\rtimes\mathbb{Z}/2}(S^A) \longrightarrow R(SU(2)\rtimes\mathbb{Z}/2) \longrightarrow R(SU(2)) \longrightarrow 0$$

$$K^0_{G\rtimes\mathbb{Z}/2}(S^A) = \mathbb{Z}[z]\{q-1\} \longrightarrow \mathbb{Z}[z,q]/q^2-1 \xrightarrow{q\mapsto 1} \mathbb{Z}[z] \longrightarrow 0$$

$$K^*_{SU(2)\rtimes\mathbb{Z}/2}(SU(2)) = \mathbb{Z}[z,q]/q^2-1 \oplus \mathbb{Z}[z]$$

## Rep. Th. Interpretation:

for $K_G(G) = F(G_+, K_G)$

Finite rep. of $LG$: factors through eval. at fin. map pts

$$
\begin{tikzcd}
LG \arrow[rr] \arrow[dr, dashed, "ev_{x_1} \times \cdots \times ev_{x_n}"'] & & GL(V) \\
& G^n \arrow[ur, dashed] &
\end{tikzcd}
$$

Rep. Space of $\Gamma$: for given, fin, dim. $V$

$$Rep(\Gamma, V) = Maps(\Gamma \to GL(V))$$

Top Cat $C(\Gamma)$:

Obj: $\coprod_V \mathrm{Rep}(\Gamma, V)$

Mor: $\coprod_{V,W} \mathrm{Rep}(\Gamma, V) \times GL(V,W)$

classifying sp. $= \mathrm{Rep}(\Gamma)$

$\mathrm{Rep}_0(\Gamma)$ = use finite reps.

<u>Th</u>. group completion of $E_\infty$-space $\mathrm{Rep}_0(LG)$ is $\simeq$ to $\underbrace{K_G(G)_0}_{\text{infinite loop sp}}$

for $K_{G\rtimes\mathbb{Z}/2}(G)$: $K_{G\rtimes\mathbb{Z}/2}(G) = F(G_+, K_{G\rtimes\mathbb{Z}/2})$ $\quad \omega : S^1 \to G,$

$LG\rtimes\mathbb{Z}/2$ : $\mathbb{Z}/2$ acts by $\alpha$ on target

or by $\alpha$ on target & reverse loops

form $C_0(LG\rtimes\mathbb{Z}/2)$

$\mathrm{Rep}_0(LG\rtimes\mathbb{Z}/2)$

Th. $K_{G\rtimes\mathbb{Z}/2}(G)_0 \simeq \mathrm{Rep}_0(LG\rtimes\mathbb{Z}/2)$ (w/ group completion)

(go through "Hochschild homology cx."

simplicial cat whose n-th level

is $LG\times G^n$

$LG \xrightarrow{\omega} G \circlearrowleft \alpha$ ← simp. cat

$LG \xrightarrow{\omega^{-1}} G \circlearrowleft \alpha$ ← simp cat. w/ involution

reverses $\{0, \dots n\}$