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"Towards the equivariant Dyer-Lashof algebras"

5/30/15

Let G be a finite group

"Equivariant" = "with/respecting a G-action"

X a G-space $\longleftrightarrow$ $G \to \mathrm{Homeo}(X)$

<u>Top</u>: objects: all G-spaces of G on X
Maps: all maps w/ conjugation action

$$\mathrm{Map}(X,Y) \circlearrowleft G$$
$$f \mapsto gfg^{-1}$$

Top^G = G-spaces & equivariant maps $\mathrm{Map}(X,Y)^G$

If $H \leq G$, then
$$\begin{tikzcd} \mathrm{Top}^G \arrow[r, "i_H^*"] & \mathrm{Top}^H \arrow[l, bend right, "G\times_H(-)"'] \arrow[l, bend left, "\mathrm{Map}_H(G,-)"] \end{tikzcd}$$

In alg, left and right adjoint are isomorphic
<u>Not</u> true in spaces

$H = \{e\} \leq G$ $\qquad$ $G \times_{\{e\}} X$ $\qquad$ This is disconnected

What are $[S^k, G \times_{\{e\}} X]^G = [S^k, (G \times_{\{e\}} X)^G]$

trivial action $\qquad$ $= \emptyset$

$$[S^k, \mathrm{Map}_{\{e\}}(G,X)]^G \cong [\iota^*_{\{e\}} S^k, X]^{\{e\}} \cong \pi_k X$$

Stabilize by inverting all representation spheres

In $\mathcal{S}p^G$ = stabilized G-spaces, have $G_+ \wedge_H X \xrightarrow{\sim} \mathrm{Map}_H(G_+, X)$

In $\mathcal{T}op^G$ homotopy groups extend to functors

$$\text{finite G-sets} = (\mathcal{S}et^G)^{op} \longrightarrow Ab$$

$$T \longrightarrow [T_+ \wedge S^k, X]^G =: \underline{\pi}_k(X)(T)$$

Have functors Σ^∞ $\mathcal{T}op^G \rightleftarrows \mathcal{S}p^G : \Omega^\infty$

$$\Omega^\infty \Sigma^\infty (X) = \varinjlim \Omega^{V_n} \Sigma^{V_n} X$$

V_n is a seq of rep that eventually contains all finite dim'l reps

$\Omega^\infty\Sigma^\infty(X)$ has an action of an operad $\mathcal{D}$

Standard etc

$\mathcal{D}(V_n)_k =$ Embeddings of $\coprod_k D(V_n)$ into $D(V_n)$

Dyer-Lashoff: operations in homology of an infinite loop space

$\mathcal{D}_k =$ Embeddings of $\coprod_k D$ into D ← little disk in $\oplus V_n$

has a $G \times \Sigma_k$-action

$\Lambda \subseteq G \times \Sigma_k$ $\qquad \mathcal{D}_k^\Lambda \simeq \begin{cases} \emptyset & \Lambda \cap \Sigma_k \neq \{e\} \\ * & \Lambda \cap \Sigma_k = \{e\} \end{cases}$

$\Rightarrow$ If $\Lambda \cap \Sigma_k = \{e\}$ have a unique map $G \times \Sigma_k / \Lambda \to \mathcal{D}_k$

$$G \times \Sigma_k/\Lambda = (G \times \Sigma_k) \times_\Lambda *$$

If $Y = \Omega^\infty \Sigma^\infty (X)$, then have $\mathcal{D}_k \times_{\Sigma_k} Y^{\times k} \to Y$

$\Rightarrow G \times \Sigma_k / \Lambda \times_{\Sigma_k} Y^k \to Y$

$\Lambda \cap \Sigma_k = \{e\} \Rightarrow \exists\, H \le G$ and $f: H \to \Sigma_k$ s.t.

$$\Lambda = \Gamma_f = \{(h, f(h)) \mid h \in H\}$$

Thus $\Lambda/_{\text{conj}} \longleftrightarrow H$ set structures on $\{1, \dots, k\}$ T_Λ

$$(G \times \Sigma_k/\Lambda) \times_{\Sigma_k} Y^{\times k} \cong G \times_H \mathrm{Map}(T, \iota_H^* Y)$$ i.e. $\mathcal{D}$-action

gives maps $G \times_H \mathrm{Map}(T, i_H^* Y) \longrightarrow Y$

If $H = G$ $\mathrm{Map}(T, Y) \to Y$ Apply $\underline{\Pi}_k$

Rmk

$$G \times_H (H \times \Sigma_k/\Lambda) \cong (G \times \Sigma_k)/\Lambda$$

$$\begin{tikzcd}
\left(S \mapsto [(S \times T)_+ \wedge S^{k'}, Y]^G\right) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \underline{\Pi}_{k'}(Y) \\
\underline{\Pi}_{k'}(Y)_T \arrow[ur] &
\end{tikzcd}$$

$G = C_2$ claim data is same data as Mackey functor $\underline{M}_{C_2}$

coefficient system $\underline{M}$

$$\begin{tikzcd}
M(C_2/C_2) \arrow[d, "\rho"] & M(C_2/e) \arrow[l] \arrow[d, "e = \mathrm{diag}"] \\
M(C_2/e) \arrow[loop below, "\gamma"] & M(C_2/e)^{\oplus 2} \arrow[l] \arrow[loop below, "\mathrm{swap}"]
\end{tikzcd}$$

This recovers $\text{res} \circ \text{trans} = \sum_{\text{weil}}$

This is double coset formula

If $T = * \amalg *$

$$\begin{tikzcd} \mathrm{Map}(* \amalg *, Y) \cong Y \times Y \arrow[d] \arrow[dl, phantom] \\ Y \end{tikzcd}$$

$$Y \times Y \xrightarrow{m} Y \quad \text{mult}$$

$\text{mult} \circ \text{swap} = \text{mult}$ in homotopy

$\Rightarrow \mathcal{D}$ algebra has mult m and maps $M(G/H, Y) \longrightarrow Y$
"transfer"

· Replace Y by cat of $\underline{\mathrm{Top}_G}$ gives model for G spectra
(Guillou, Merling, Osorno)

Bredon homology $Y \mapsto H_*(Y, \underline{M})$
$* \in RO(G)$

Dyer-Lashoff algebra: $H_*(\mathcal{D}_k/\Sigma_k; \underline{M})$ if $\underline{M}$ is rational then have alg descriptn

$\mathbb{Q}$ DL algebra is generated by m & all transfers

Need $H_*(-; \underline{M})$ to be represented by a commutative ring Spectrum.

IF M is not rat'l and $|G|$ not invertible then get a copy of traditional Dyer-Lashoff alg for each? Now get odd Adem relations.

Motivic theory missing Ω^∞

Po Hu give way of tensoring with a space.

Q: (Tony Elmendorf) Missing classically that Homology of Spaces in $\mathcal{D}_k$ are homology of Σ_k

A: Yes. Representations of Σ_k in Mackey Functors have wrong notion of projective

Need resolution of Burnside Mackey functor

- every rat'l Mackey functor is projective