HHR-Towards the equivariant Dyer-Lashof algebras

Note Title 5/30/2015

Let G be a finite gp

Define G-action, G-space, G-map

$\underline{Top}$ = category of G-spaces and cont maps
$Top(X,Y)$ has conjugation action

Top^G = cat of G-spaces + equiv maps.

If $H \subset G$ then we have functors

$$\begin{tikzcd} Top^G \arrow[rr, "i_H^*"] & & Top^H \arrow[ll, bend right, "G\times_H(-)"'] \arrow[ll, bend left, "Map_H(G,-)"] \end{tikzcd}$$

left adjoint = induction

right adjoint = coinduction

For $G=\{e\}$, we get $|G|$ copies of X for left adj and $X^{|G|}$ for right adjoint

$$[S^k, G\times_{\{e\}} X]^G = [S^k, (G\times_{\{e\}} X)^G] = \text{empty}$$

$$[S^k, Map(G,X)]^G = [S^k, X]^{\{e\}} = \pi_k X$$

Stabilize by inverting all rep spheres S^V

This leads to a category Sp^G we have

$G_+ \times_H X \longrightarrow Map_H(G_+, X)$ is an equivalence

In Top^G, htpy gps extend to functors

$$\left\{\begin{array}{c}\text{finite}\\ G\text{-sets}\end{array}\right\} = (Set^G)^{op} \longrightarrow Ab$$

$$T \longmapsto [T_+ \wedge S^k, X]^G =: \underline{\pi}_k(X)(T)$$

We have functors

$$\Sigma^\infty : Top^G \rightleftarrows Sp^G : \Omega^\infty$$

$$\Omega^\infty \Sigma^\infty X = \operatorname{colim}_n \Omega^{V_n} \Sigma^{V_n} X$$ where $\{V_n\}$ is a seq of reps containing all finite reps

This has an action of an operad $\mathcal{D}$ (little disks)

where $(\mathcal{D}(V_n))_k$ = Embeddings of

$\coprod_k D(V_n)$ into $D(V_n)$.

$\mathcal{D}_k$ = Embeddings $\coprod_k D$ into D (little disk in $\varinjlim V_n$)

It has an action of $G \times \Sigma_k$. For $\Lambda \subset G \times \Sigma_k$,

$$\mathcal{D}_k^{\Lambda} \simeq \begin{cases} \emptyset & \text{if } \Lambda \cap \Sigma_k \neq \{e\} \\ * & \text{if } \Lambda \cap \Sigma_k = \{e\} \end{cases}$$

$\Rightarrow$ If $\Lambda \cap \Sigma_k = \{e\}$, we have a unique (up to htpy) map $G \times \Sigma_k / \Lambda \longrightarrow \mathcal{D}_k$

$$\begin{gathered} G \times \Sigma_k / \Lambda \\ \| \\ G \times_{\Sigma_k} X_{pt}^{pt} \end{gathered}$$

If $Y = \Omega^\infty \Sigma^\infty X$, then we have structure maps

$$\mathcal{D}_k \times_{\Sigma_k} Y^{\times k} \longrightarrow Y$$

and hence $G \times \Sigma_k / \Lambda \times_{\Sigma_k} Y \longrightarrow Y$

For $\Lambda \cap \Sigma_k = \{e\}$, $\exists H \leq G$ and $f : H \to \Sigma_k$ s.t.

$\Lambda = \Gamma_f = \{(h, f(h)) \mid h \in H\}$ i.e.

$\Lambda \longleftrightarrow$ H-set structures^T on $\{1, 2, \cdots k\}$

$$G \times \Sigma_k / \Lambda \times_{\Sigma_k} Y^{\times k}$$

$$\cong$$

$G \times_H \mathrm{Map}(T, i_H^* Y)$ i.e. $\mathcal{O}$-action gives maps

$$G \times_H \mathrm{Map}(T, i_*^H Y) \longrightarrow Y$$

For $H = G$ this gives $\mathrm{Map}(T, Y) \longrightarrow Y$ and hence

$$\left(S \mapsto [(S \times T)_n, S^n, Y]^G\right) \longrightarrow \underline{\pi}_n(Y)$$

$$\cong$$

$$\underline{\pi}_n(Y)(T)$$

This leads to a **Mackey functor.**

Let $G = C_2$ and M a **Mackey functor**

$$\underline{M} = \begin{tikzcd} M(C_2/C_2) \arrow[d, "\rho"'] \arrow[d, phantom, shift left=6, "\text{restriction}"] & M(C_2/e) \arrow[l, "\text{transfer}"'] \arrow[d, "\rho"] \\ M(C_2/e) \arrow[loop below, "\gamma"] & M(C_2/e)^{\oplus 2} \arrow[l] \arrow[loop below, "\text{map}"] \end{tikzcd}$$

For $T = * \amalg *$, $\quad \mathrm{Map}(T, Y) \cong Y \times Y$

$$\begin{tikzcd} \mathrm{Map}(T, Y) \arrow[d, "\text{structure map}"'] & Y \times Y \arrow[dl, "m"] \\ Y & \end{tikzcd}$$

htpy comm mult.

Bredon homology $\quad Y \longmapsto H_\star(Y;\underline{M})$

$\underline{M}$ a Mackey functor. for $\star \in RO(G)$.

Dyer-Lashof algebra $\quad H_\star(D_k/\Sigma_k;\underline{M})$

For $\underline{M}$ rational, we have an alg description

$\mathbb{Q}$ DL algebra is generated by the mult m and all restrictions.

We need $\underline{M}$ to be a Tambara functor.