

$X$  smooth complex alg variety

$X^{an}$  cplx mfld

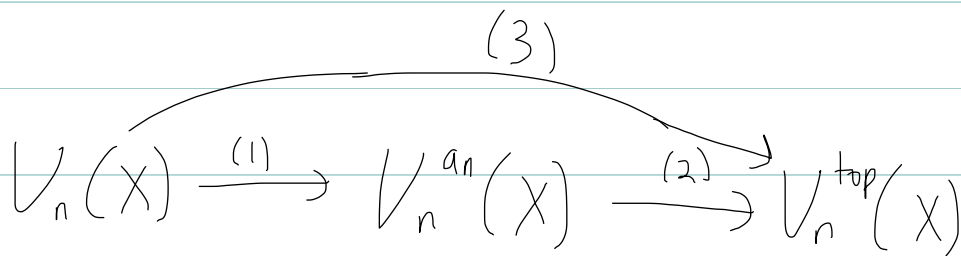
"algebraizing top vector bundles"

5/30/15

$V_n(X) =$  iso classes of rk  $n$  v.b. on  $X$

$V_n^{an}(X) =$  iso classes rk  $n$  holo v.b. on  $X^{an}$

$V_n^{top}(X) =$  ———  $\mathbb{C}$ -top v.b.



Def.  $V \in \text{Im}(3)$  is  $V$  "algebraizable"

Q. Can we characterize algebraizable vector bundles?

(3) is not in general injective for  $X$  a curve  
 surjectivity ——— a surface

If  $E$  is a top vector bundle  $c_i^{top} \in H^{2i}(X, \mathbb{Z})$   
 an alg vector bundle  $c_i \in CH^i(X) =$  Chow group of codim  
 cycles mod  
 rational equiv

$$CH^i(X) \longrightarrow H^{2i}(X, \mathbb{Z}) \quad \text{cycle class map}$$

a necessary condition in algebraizability is that top chern classes lie in image of cycle class map.

Q: If a top vector bundle has alg Chern classes can the bundle be algebraized?

Two extreme cases

$X$  smooth, proper alg variety (1) is a bijection  
GAGA

has htpy type of a CWplx  
of  $\dim = 2 \dim X$

$X$  smooth affine alg variety

(2) is a bijection  
Grauert's Oka-principle  
(h-principle)

has the htpy type of a CW complex  
of  $\dim \leq \dim X$

$X$  a curve every top vector bundle is algebraizable

$X$  a surface: if <sup>topological</sup>  $\wedge$  Chern classes are alg then the bundle is algebraizable

if  $X$  proper Schwarzenberger '60s

if  $X$  affine Murthy-Swan '60's

Atiyah-Rees

In  $\dim \geq 3$ ,  $\mathbb{P}^3$  every top vector bundle is algebraic '76

rationaly connected 3-fold '87

Banica-Putinar

For smooth affine things, every top vector bundle w/ alg Chern classes is algebraizable

Mohan-Kumar Murthy '82

joint with M. Hopkins, J. Fasel

Thm (A. Fasel Hopkins) Question has a negative answer for smooth affine varieties of  $\dim \geq 4$

• If  $X$  is a smooth complex affine 4-fold,  $\mathcal{E}$  a rk 2 top vector bundle on  $X$  with Chern classes  $c_1^{\text{top}}, c_2^{\text{top}}$  then  $\mathcal{E}$  admits an alg structure  $\Leftrightarrow$

$\exists c_i \in H^i(X)$   $i=1,2$  s.t.  $c_i$  map to  $c_i^{\text{top}}$  under the cycle class map &

$$Sq^2 c_2 + c_1 \cup c_2 = 0 \in H^3(X)/2$$

• There exists a smooth, complex, affine 4-fold s.t.

$$H^i(X) \rightarrow H^{2i}(X, \mathbb{Z}) \text{ is an iso } i \leq 2$$

& s.t.  $Sq^2 c_2 \neq 0$ .

the existence of such a bundle is related to the failure of the integral Hodge conjecture

How does one prove such things?

$R$  infinite perfect

Thm Let  $X$  be  $S_m$ , affine /  $R$ .  $[X, BGL_n]_{\mathbb{A}^1} \cong V_n(X)$

If BGL<sub>n</sub> did represent  $V_n(X)$  then

$$\underline{\text{Ex}}: (1) X \times \mathbb{A}^1 \rightarrow X$$

$$V_n(X) \rightarrow V_n(X \times \mathbb{A}^1) \quad \text{should be } \cong$$

not true

$$(2) X = \mathbb{A}^n \quad \text{Quillen-Suslin '76}$$

$$(3) Q_4 = \{x_1, x_2 + x_3 x_4 = z(z+1)\} = TS_4$$

$$E_2 \subset \{x_1 = x_3 = 0, z = -1\}$$

$$X_4 = Q_4 \setminus E_2$$

$$\begin{array}{ccc} \mathbb{A}^5 & \longrightarrow & X_4 \\ \uparrow & \text{Zariski locally trivial w/ fibers } \mathbb{A}^1 & \\ \mathbb{A}^1 & & \end{array}$$

There is a non-trivial alg bundle on  $Q_4$

A, B. Doran

$$\Rightarrow Q_4 \setminus E_4 \text{ —————}$$

Define  $\mathcal{H}(k)$  unstable  $A^1$ -htpy cat

Idea: understand our original v.b. question using obstruction theory

$$\begin{array}{ccc} BGL_2 & \xrightarrow{(c_1, c_2)} & K(\mathbb{Z}(1), 2) \times K(\mathbb{Z}(2), 4) \\ \uparrow & & \downarrow \\ \mathcal{H} & & K(\mathbb{K}_1^M, 1) \times K(\mathbb{K}_2^M, 2) \end{array} \quad \text{Eilenberg-MacLane Spaces}$$