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"algebraizing top vector bundles"

5/30/15

X smooth complex alg variety

X^{an} cplx mfld

$V_n(X) =$ iso classes of rk n v.b. on X

$V_n^{an}(X) =$ iso classes rk n holo v.b. on X^{an}

$V_n^{top}(X) =$ —— $\mathbb{C}$-top v.b.

$$\begin{tikzcd} V_n(X) \arrow[r, "(1)"] \arrow[rr, bend left, "(3)"] & V_n^{an}(X) \arrow[r, "(2)"] & V_n^{top}(X) \end{tikzcd}$$

<u>Def:</u> $V \in \mathrm{Im}(3)$ is V "algebraizable"

<u>Q:</u> Can we characterize algebraizable vector bundles?

(3) is not in general injective for X a curve
surjectivity —— a surface

If E is a top vector bundle $c_i^{top} \in H^{2i}(X, \mathbb{Z})$
an alg vector bundle $c_i \in CH^i(X) =$ Chow group of codim i cycles modulo rat'l equiv

$$CH^i(X) \longrightarrow H^{2i}(X, \mathbb{Z}) \quad \text{cycle class map}$$

a necessary condition in algebraizibility is that top chern classes lie in image of cycle class map.

Q: If a top vector bundle has alg Chern Classes can the bundle be algebraized?

Two extreme cases

X smooth, proper alg variety (1) is a bijection
GAGA

has htpy type of a CW cplx of dim $= 2 \dim X$

X smooth affine alg variety (2) is a bijection
Grauert's Oka-principle
(h-principle)

has the htpy type of a CW complex of dim $\leq \dim X$

X a curve every top vector bundle is algebraizable

X a surface: if topological Chern classes are alg then the bundle is algebraizable

if X proper Schwarzenberger '60s

if X affine Murthy-Swan '60's

In dim ≥ 3, $\mathbb{P}^3$ every top vector bundle is algebraic — Atiyah-Rees '76

rationally connected 3-fold '87

Banica-Putinar

For smooth affine things, every top vector bundle w/ alg Chern classes is algebraizable

Mohan-Kumar Murthy '82

joint with M. Hopkins, J. Fasel

<u>Thm</u> (A. Fasel Hopkins) Question has a negative answer for smooth affine varieties of $\dim \geq 4$

• If X is a smooth complex affine 4-fold, $\mathcal{E}$ a rk 2 top vector bundle on X with Chern classes c_1^{top}, c_2^{top} then $\mathcal{E}$ admits an alg structure $\Leftrightarrow$

$\exists\ c_i \in CH^i(X)$ $i=1,2$ s.t. c_i map to c_i^{top} under the cycle class map &

$$Sq^2 c_2 + c_1 \cup c_2 = 0 \in CH^3(X)/2$$

• There exists a smooth, complex, affine 4-fold s.t.

$$CH^i(X) \to H^{2i}(X, \mathbb{Z}) \text{ is an iso } i \leq 2$$

& s.t. $Sq^2 c_2 \neq 0$.

the existence of such a bundle is related to the failure of the integral Hodge conjecture

How does one prove such things?

R infinite perfect

<u>Thm</u> Let X be sm, affine $/R$. $[X, BGL_n]_{\mathbb{A}^1} \cong V_n(X)$

If BGL_n did represent $V_n(X)$ then

Ex: (1) $X \times \mathbb{A}^1 \to X$

$V_n(X) \to V_n(X \times \mathbb{A}^1)$ should be $\cong$

not true

(2) $X = \mathbb{A}^n$ Quillen-Suslin '76

(3) $Q_4 = \{x_1 x_2 + x_3 x_4 = z(z+1)\} = TS_4$

$E_2 \subset \{x_1 = x_3 = 0, \ z = -1\}$

$X_4 = Q_4 \setminus E_2$

$$\begin{array}{ccl} \mathbb{A}^5 & \longrightarrow & X_4 \\ \uparrow & & \text{Zariski locally trivial w/ fibers } \mathbb{A}^1 \\ \mathbb{A}^1 & & \end{array}$$

There is a non-trivial alg bundle on Q_4

A, B. Doran

$\Rightarrow Q_4 \setminus E_4$ ———

Define $\mathcal{H}(k)$ unstable $\mathbb{A}^1$-htpy cat

Idea: understand our original v.b. question using obstruction thy

$$\begin{tikzcd} BGL_2 \arrow[r, "(c_1, c_2)"'] & K(\mathbb{Z}(1),2) \times K(\mathbb{Z}(2),4) \arrow[d] \\ ? \arrow[u] & K(\underline{K}^M_1, 1) \times K(\underline{K}^M_2, 2) \end{tikzcd}$$

Eilenberg-MacLane spaces