5.7 Problems

Find a fundamental matrix of each of the systems in Problems 1 through 8, then apply Eq. (8) to find a solution satisfying the given initial conditions.

1. $\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

2. $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

3. $\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

4. $\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

5. $\mathbf{x}' = \begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

6. $\mathbf{x}' = \begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

7. $\mathbf{x}' = \begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

8. $\mathbf{x}' = \begin{bmatrix} 3 & 2 & 2 \\ -5 & -4 & -2 \\ -5 & 5 & 3 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Compute the matrix exponential $e^{\mathbf{A}t}$ for each system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ given in Problems 9 through 20.

9. $x_1' = 5x_1 - 4x_2,\ x_2' = 2x_1 - x_2$

10. $x_1' = 6x_1 - 6x_2,\ x_2' = 4x_1 - 4x_2$

11. $x_1' = 5x_1 - 3x_2,\ x_2' = 2x_1$

12. $x_1' = 5x_1 - 4x_2,\ x_2' = 3x_1 - 2x_2$

13. $x_1' = 9x_1 - 8x_2,\ x_2' = 6x_1 - 5x_2$

14. $x_1' = 10x_1 - 6x_2,\ x_2' = 12x_1 - 7x_2$

15. $x_1' = 6x_1 - 10x_2,\ x_2' = 2x_1 - 3x_2$

16. $x_1' = 11x_1 - 15x_2,\ x_2' = 6x_1 - 8x_2$

17. $x_1' = 3x_1 + x_2,\ x_2' = x_1 + 3x_2$

18. $x_1' = 4x_1 + 2x_2,\ x_2' = 2x_1 + 4x_2$

19. $x_1' = 9x_1 + 2x_2,\ x_2' = 2x_1 + 6x_2$

20. $x_1' = 13x_1 + 4x_2,\ x_2' = 4x_1 + 7x_2$

In Problems 21 through 24, show that the matrix $\mathbf{A}$ is nilpotent and then use this fact to find (as in Example 3) the matrix exponential $e^{\mathbf{A}t}$.

21. $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

22. $\mathbf{A} = \begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix}$

23. $\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

24. $\mathbf{A} = \begin{bmatrix} 3 & 0 & -3 \\ 5 & 0 & 7 \\ 3 & 0 & -3 \end{bmatrix}$

Each coefficient matrix $\mathbf{A}$ in Problems 25 through 30 is the sum of a nilpotent matrix and a multiple of the identity matrix. Use this fact (as in Example 6) to solve the given initial value problem.

25. $\mathbf{x}' = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

26. $\mathbf{x}' = \begin{bmatrix} 7 & 0 \\ 11 & 7 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$